Sufficient Conditions for Zeno Behavior in Lagrangian Hybrid Systems

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Abstract. This paper presents easily verifiable sufficient conditions for the existence of Zeno behavior in Lagrangian hybrid systems, i.e., hybrid systems modeling mechanical systems undergoing impacts.

1 Introduction

This paper is motivated by the lack of analytic tools for proving the existence of Zeno behavior in nontrivial hybrid systems. In particular, mechanical systems undgergoing impacts, modeled by Lagrangian hybrid systems [3], provide a large class of systems that often appear to display Zeno behavior. While Zeno behavior is often intuitively clear and supported with simulation results [2], formal proofs of Zeno behavior have been limited to very simple systems, e.g., the bouncing ball.

To study Zeno behavior, we consider *Zeno equilibria*—subsets of the continuous domains of a hybrid system that are fixed points of the discrete dynamics but not the continuous dynamics—which are defined in analogy to equilibria of dynamical systems. Given the success of studying isolated equilibria in dynamical systems, a natural starting point to the study of Zeno behavior is a detailed analysis of *isolated Zeno equilibria*—those Zeno equilibria with no other nearby Zeno equilibria. Recently, however, it was observed that Lagrangian hybrid systems with isolated Zeno equilibria must have one dimensional configuration manifolds [6]. Most Lagrangian hybrid systems of interest, however, have higher dimension configuration manifolds. Thus a large set of systems believed to show Zeno behavior cannot be adequately studied with attention restricted to isolated Zeno equilibria.

These observations motivate the main result of this paper: sufficient conditions for Zeno behavior in Lagrangian hybrid systems with configuration spaces of arbitrary dimension. These conditions for Lagrangian hybrid systems generalize those in [6], but remain remarkably simple. When applied to examples, such as a ball bouncing on a sinusoidal surface or a pendulum on a cart, the conditions for Zeno behavior are easily verifiable and intuitively appealing.

This work complements other work on Zeno, including [7], [4] and [5].

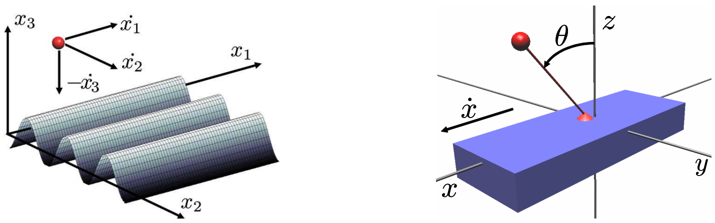

Fig. 1. Ball bouncing on a sinusoidal surface (left). Pendulum on a cart (right).

2 Simple Hybrid Mechanical Systems

Mechanical systems undergoing impacts are naturally modeled as hybrid systems. In this section, we will consider hybrid systems of this form and recall how one obtains such systems from hybrid Lagrangians, which are the hybrid analogue of Lagrangians. For more on hybrid Lagrangians and Lagrangian hybrid systems, see [1].

Due to space constraints, we are unable to formally define hybrid systems, executions and Zeno equilibria. We will use the definitions and notation from [6] unchanged to avoid any confusion.

Hybrid Lagrangians and Lagrangian Hybrid Systems. In the context of smooth mechanical systems, one begins with a Lagrangian $L : T\Theta \to \mathbb{R}$ on a configuration space Θ and associates to this Lagrangian a dynamical system:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + N(\theta) = 0,$$

through the Euler-Lagrange equations. Similarly, in the context of mechanical systems undergoing impacts, one begins with a *hybrid Lagrangian* and associates to this a *Lagrangian hybrid system*. In particular, consider the following:

Definition 1. *A hybrid Lagrangian is a tuple* $\mathbf{L} = (\Theta, L, h)$*, where* $\Theta \subset \mathbb{R}^n$ *is the configuration space,* $L : T\Theta \to \mathbb{R}$ *is a Lagrangian, and* $h : \Theta \to \mathbb{R}$ *is a unilateral constraint function. We assume that* 0 *is a regular value of* h*.*

Given a hybrid Lagrangian $\mathbf{L} = (\Theta, L, h)$, the *Lagrangian hybrid system associated to* $\mathbf{L}$ is the hybrid system

$$\mathscr{H}_{\mathbf{L}} = (\Gamma = (\{q\}, \{(q,q)\}), D_{\mathbf{L}}, G_{\mathbf{L}}, R_{\mathbf{L}}, F_{\mathbf{L}}),$$

where Γ is a graph with one node and one edge, $D_{\mathbf{L}} = \{D_h\}$ and $G_{\mathbf{L}} = \{G_h\}$ are given by

$$D_h = \{(\theta,\dot{\theta}) \in T\Theta : h(\theta) \geq 0\}, \quad G_h = \{(\theta,\dot{\theta}) \in D_h : h(\theta) = 0,\ dh(\theta)\dot{\theta} \leq 0\},$$

$F_{\mathbf{L}} = \{f_L\}$ is the vector field obtained from the Lagrangian L, and $R_{\mathbf{L}} = \{R_h\}$ with $R_h(\theta,\dot{\theta}) = (\theta, P(\theta,\dot{\theta}))$, where

$$P(\theta,\dot{\theta}) = \dot{\theta} - (1+e)\frac{dh(\theta)\dot{\theta}}{dh(\theta)M(\theta)^{-1}dh(\theta)^T}M(\theta)^{-1}dh(\theta)^T, \tag{1}$$

with coefficient of restitution $0 \leq e \leq 1$. Zeno equilibria of Lagrangian hybrid systems are exactly the fixed points of R_h. More details on this construction can be found in [6].

Examples. We now present two examples that will be considered throughout the rest of the paper in order to illustrate the concepts involved.

Example 1 (Ball). Our first running example is a ball bouncing on a sinusoidal surface (cf. Fig. 1). In this case $\mathbf{B} = (\Theta_{\mathbf{B}}, L_{\mathbf{B}}, h_{\mathbf{B}})$, where $\Theta_{\mathbf{B}} = \mathbb{R}^3$, and for $x = (x_1, x_2, x_3)$,

$$L_{\mathbf{B}}(x, \dot{x}) = \frac{1}{2} m \|\dot{x}\|^2 - mgx_3, \qquad h_{\mathbf{B}}(x_1, x_2, x_3) = x_3 - \sin(x_2).$$

From this hybrid Lagrangian, one obtains a Lagrangian hybrid system $\mathscr{H}_{\mathbf{B}}$.

Example 2 (Cart). Our second running example is a constrained pendulum on a cart (cf. Fig. 1); this is a variation on the classical pendulum on a cart, where the pendulum is not allowed to "pass through" the cart. In this case $\mathbf{C} = (\Theta_{\mathbf{C}}, L_{\mathbf{C}}, h_{\mathbf{C}})$, where $\Theta_{\mathbf{C}} = \mathbb{S}^1 \times \mathbb{R}$, $q = (\theta, x)$, and

$$L_{\mathbf{C}}(\theta, \dot{\theta}, x, \dot{x}) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{x} \end{pmatrix} \begin{pmatrix} mR^2 & mR\cos(\theta) \\ mR\cos(\theta) & M + m \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{x} \end{pmatrix} - mgR\cos(\theta).$$

where m is the mass of the pendulum, M is the mass of the cart and R is the length of the pendulum. Finally, the constraint $h_{\mathbf{C}}(\theta, x) = \cos(\theta)$ ensures that the pendulum cannot pass through the cart. One obtains a Lagrangian hybrid system $\mathscr{H}_{\mathbf{C}}$ from the hybrid Lagrangian $\mathbf{C}$.

3 Sufficient Conditions for Zeno Behavior in Lagrangian Hybrid Systems

In this section, we present sufficient conditions for the existence of Zeno behavior in Lagrangian hybrid systems. Before presenting this conditions, we characterize Zeno equilibria in systems of this form.

Zeno equilibria in Lagrangian hybrid systems. If $\mathscr{H}_{\mathbf{L}}$ is a Lagrangian hybrid system, then due to the special form of these systems we find that the point $z = \{(\theta^*, \dot{\theta}^*)\}$ is a Zeno equilibria iff $\dot{\theta}^* = P(\theta, \dot{\theta}^*)$, with P given in (1). In particular, the special form of P implies that this holds iff $dh(\theta^*)\dot{\theta}^* = 0$. Therefore the set of all Zeno equilibria for a Lagrangian hybrid system is given by the hypersurfaces in G_h:

$$Z = \{(\theta, \dot{\theta}) \in G_h : dh(\theta)\dot{\theta} = 0\}.$$

Note that if $\dim(\Theta) > 1$, the Zeno equilibria in Lagrangian hybrid systems are always non-isolated (see [6])—this motivates the study of such equilibria.

Theorem 1. *Let $\mathscr{H}_{\mathbf{L}}$ be a Lagrangian hybrid system and Let $z = \{(\theta^*, \dot{\theta}^*)\}$ be a Zeno equilibria of $\mathscr{H}_{\mathbf{L}}$. If $0 < e < 1$ and*

$$\ddot{h}(\theta^*, \dot{\theta}^*) = (\dot{\theta}^*)^T H(h(\theta^*))\dot{\theta}^* + dh(\theta^*)M(\theta^*)^{-1}(-C(\theta^*, \dot{\theta}^*)\dot{\theta}^* - N(\theta^*)) < 0,$$

where $H(h(\theta^*))$ is the Hessian of h at θ^*, then there is a neighborhood $W \subset D_h$ of $(\theta^*, \dot{\theta}^*)$ such that for every $(\theta, \dot{\theta}) \in W$, there is a unique Zeno execution χ of $\mathscr{H}_{\mathbf{L}}$ with $c_0(\tau_0) = (\theta, \dot{\theta})$.

Example 3 (Ball). We first demonstrate that the hybrid system $\mathscr{H}_{\mathbf{B}}$ modeling a ball bouncing on a sinusoidal surface is Zeno. First, the Zeno equilibria of this system are given by the set

$$Z = \{(x, \dot{x}) \in G_{h_{\mathbf{B}}} : \dot{x}_3 - \dot{x}_2 \cos(x_2) = 0\}.$$

Now, one can easily verify that for $(x^*, \dot{x}^*) \in Z$

$$\ddot{h}_{\mathbf{B}}(x^*, \dot{x}^*) = \sin(x_2)\dot{x}_2^2 - g.$$

Therefore, there are clearly Zeno equilibria satisfying the conditions of Theorem 1, namely when $\dot{x}_2$ is small, and thus $\mathscr{H}_{\mathbf{B}}$ is Zeno.

Example 4 (Cart). We now demonstrate that the hybrid system modeling a pendulum on a cart, $\mathscr{H}_{\mathbf{C}}$, is Zeno. First, note that the Zeno equilibria are given by the set:

$$Z = \{(\theta, x, \dot{\theta}, \dot{x}) \in G_{h_{\mathbf{C}}} : \sin(\theta)\dot{\theta} = 0\},$$

and for $(\theta^*, x^*, \dot{\theta}^*, \dot{x}^*) \in Z$,

$$\ddot{h}_{\mathbf{C}}(\theta^*, x^*, \dot{\theta}^*, \dot{x}^*) = -\frac{g}{R} < 0.$$

Therefore, for every Zeno equilibria of the pendulum on a cart there a neighborhood of the Zeno equilibria such that every execution with an initial condition in that neighborhood is Zeno.

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