Parameter Sensitivity and Boundedness of Robotic Hybrid Periodic Orbits ⋆

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Abstract: Model-based nonlinear controllers like feedback linearization and control Lyapunov functions are highly sensitive to the model parameters of the robot. This paper addresses the problem of realizing these controllers in a particular class of hybrid models–systems with impulse effects–through a *parameter sensitivity measure*. This measure quantifies the sensitivity of a given model-based controller to parameter uncertainty along a particular trajectory. By using this measure, output boundedness of the controller (computed torque+PD) will be analyzed. Given outputs that characterize the control objectives, i.e., the goal is to drive these outputs to zero, we consider Lyapunov functions obtained from these outputs. The main result of this paper establishes the ultimate boundedness of the output dynamics in terms of this measure via these Lyapunov functions under the assumption of stable hybrid zero dynamics. This is demonstrated in simulation on a 5-DOF underactuated bipedal robot.

Keywords: Hybrid Zero Dynamics, Parameter Sensitivity Measure, System Identification.

1. INTRODUCTION

Model based controllers like stochastic controllers Byl and Tedrake (2009), feedback linearization Westervelt et al. (2007), the control Lyapunov functions (CLFs) Ames et al. (2014) all require the knowledge of an accurate dynamical model of the system. The advantage of these methods are that they yield sufficient convergence for highly dynamic robotic applications, e.g., quadrotors and bipedal robots, where exponential convergence of control objectives is used to achieve guaranteed stability of the system. This is especially true of bipedal walking robots where *rapid* exponential convergence is used Ames et al. (2014). While these controllers have yielded good results when an accurate dynamical model is known, there is a need for quantifying how accurate the model has to be to realize the desired tracking error bounds. These application domains point to the need for a way to measure parameter uncertainty and a methodology to design controllers for nonlinear hybrid systems, like bipedal robots, that can converge to the control objective under parameter uncertainty.

The goal of this paper is to establish a relationship between parameter uncertainty and the output error bounds on systems with alternating continuous and discrete events, i.e., hybrid systems, while considering a specific example: bipedal walking robots. Inspired by the sensitivity functions utilized for linear systems Zhou et al. (1996), a *parameter sensitivity measure* is defined for continuous systems and the relationship between the boundedness and the measure is established through the use of Lyapunov functions. In the context of hybrid systems, along with defining the measure for the continuous event, an *impact measure* is defined to include the effect of parameter variations in the discrete event. The resulting overall sensitivity measure thus represents how sensitive a given controller is to parameter variations for hybrid systems. When described in terms of Lyapunov functions, which are constructed from the zeroing outputs of the robot, the *parameter sensitivity measure* naturally yields the ultimate bound on the outputs. Considering a 5-DOF bipedal robot, AMBER, shown in Fig. 1, where a stable periodic orbit on the hybrid zero dynamics translates to a stable walking gait on the bipedal robot, the ultimate bound on this periodic orbit will be determined through the use of a particular controller: computed torque+PD.

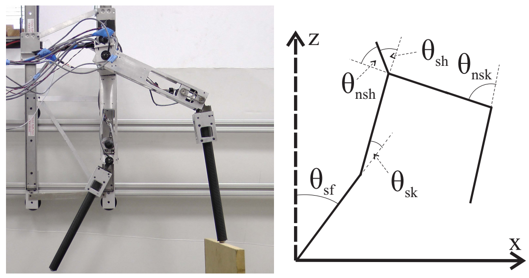

Fig. 1. The biped AMBER (left) and the stick figure of AMBER showing the configuration angles (right).

The paper is structured in the following fashion: Section 2 introduces the robot model and the control methodology used–CLFs through the method of computed torque. Section 3 assesses the controller used for the uncertain model of the robot and establishes the resulting uncertain behavior through Lyapunov functions. In Section 4, the resulting uncertain dynamics exhibited by the robot is measured formally through the construction of *parameter sensitivity measure*, which is the main formulation of this paper on which the formal results will build. It will be shown that there is a direct relationship between the ultimate bound

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on the Lyapunov function and the *parameter sensitivity measure*, which motivates the introduction of an auxiliary controller–computed torque+PD. This will be utilized for establishing bounds for the entire dynamics, under the assumption of a stable limit cycle in the zero dynamics. This method is extended to hybrid systems through the introduction of an impact measure in Section 5. Under the assumption that the hybrid zero dynamics is stable, the computed torque controller appended with the auxiliary input is applied on the model, which results in bounded dynamics of the underactuated hybrid system. The paper concludes with simulation results on a 5-DOF bipedal robot, AMBER, in Section 6.

2. ROBOT DYNAMICS AND CONTROL

A robotic model can be modeled as n-link manipulator. Given the configuration space $\mathbb{Q} \subset \mathbb{R}^n$, with the coordinates $q \in \mathbb{Q}$, and the velocities $\dot{q} \in T_q\mathbb{Q}$, the equation of motion of the n-DOF robot can be defined as:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = B\mathrm{T}, \tag{1}$$

where $D(q) \in \mathbb{R}^{n\times n}$ is the mass inertia matrix of the robot that includes the motor inertia terms, $C(q,\dot{q}) \in \mathbb{R}^{n\times n}$ is the matrix of coriolis and centrifugal forces, $G(q) \in \mathbb{R}^n$ is the gravity vector, $\mathrm{T} \in \mathbb{R}^k$ is the torque input and $B \in \mathbb{R}^{n\times k}$ is the mapping from torque to joints.

AMBER. Considering the 5-DOF underactuated bipedal robot shown in Fig. 1, the configuration can be defined as: $q = (q_{sa}, q_{sk}, q_{sh}, q_{nsh}, q_{nsk})$ corresponding to stance ankle (sa), stance and non-stance knee (sk,nsk), stance and non-stance hip angles (sh,nsh) of the robot. Since the ankle is not actuated, the number of actuators is $k = 4$.

Outputs. We will utilize the method of computed torque since it is widely used in robotic systems. It is also convenient in the context of uncertain models which will be considered in the next section. To realize the controller, *outputs* are picked which are functions of joint angles referred to as actual outputs $y_a : \mathbb{Q} \to \mathbb{R}^k$, which are made to track functions termed the desired outputs $y_d : \mathbb{Q} \to \mathbb{R}^k$. The objective is to drive the error $y(q) = y_a(q) - y_d(q) \to 0$. These outputs are also termed *virtual constraints* in Westervelt et al. (2007). The outputs are picked such that they are relative degree two outputs (see Sastry (1999)). Given the output y:

$$\ddot{y} = \underbrace{\frac{\partial y}{\partial q}}_{J}\,\ddot{q} + \underbrace{\dot{q}^T\frac{\partial^2 y}{\partial q^2}}_{\dot{J}}\,\dot{q}. \tag{2}$$

Since, $k < n$, we include $n-k$ rows to J and $\dot{J}$ to make the co-efficient matrix of $\ddot{q}$ full rank. These rows correspond to the configuration which are underactuated resulting in:

$$\begin{bmatrix} 0 \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} D_1 \\ J \end{bmatrix}\ddot{q} + \begin{bmatrix} H_1 \\ \dot{J}\dot{q} \end{bmatrix}, \tag{3}$$

where H_1 is the $n-k$ rows of $H(q,\dot{q}) = C(q,\dot{q})\dot{q}+G(q)$, and D_1 is the $n-k$ rows of the expression, $D(q)$. It should be observed that since the underactuated degrees of freedom have zero torque being applied, the resulting EOM of the robot leads to 0 on the left hand side of (3), and hence the choice of rows. Accordingly, we can define the desired acceleration for the robot to be:

$$\ddot{q}_d = \begin{bmatrix} D_1 \\ J \end{bmatrix}^{-1}\left(\begin{bmatrix} 0 \\ \mu \end{bmatrix} - \begin{bmatrix} H_1 \\ \dot{J}\dot{q} \end{bmatrix}\right), \tag{4}$$

where μ is a linear control input. The resulting torque controller that realizes this desired acceleration in the robot can be defined as:

$$B\mathrm{T} = D(q)\ddot{q}_d + C(q,\dot{q})\dot{q} + G(q). \tag{5}$$

Substituting (5) and (4) in (1) results in linear dynamics: $\ddot{y} = \mu$, with μ chosen through a CLF based controller.

Zero Dynamics and CLF. If we define the vector: $\eta = [y^T, \dot{y}^T]^T$, the dynamics can be reformulated as:

$$\dot{\eta} = \underbrace{\begin{bmatrix} 0_{k\times k} & 1_{k\times k} \\ 0_{k\times k} & 0_{k\times k} \end{bmatrix}}_{F}\eta + \underbrace{\begin{bmatrix} 0_{k\times k} \\ 1_{k\times k} \end{bmatrix}}_{G}\mu, \tag{6}$$

which represent the controllable dynamics of the system. Since, $k < n$ there are states that are not directly controllable which represent the zero dynamics of the system and can be expressed as:

$$\dot{z} = \Psi(\eta, z), \tag{7}$$

where $z \in Z \subseteq \mathbb{R}^{2(n-k)}$ is the zero dynamic coordinates of the system (see Westervelt et al. (2007)).

Consider the Lyapunov Function: $V(\eta) = \eta^T P\eta$, where P is the solution to the continuous-time algebraic Riccati equation (CARE). Taking the derivative yields:

$$\dot{V}(\eta) = \eta^T(F^TP + PF)\eta + 2\eta^TPG\mu. \tag{8}$$

To find a specific value of μ, we can utilize a minimum norm controller (see Freeman and Kokotovic (2008)) which minimizes $\mu^T\mu$ subject to the inequality constraint:

$$\dot{V} = \eta^T(F^TP + PF)\eta + 2\eta^TPG\mu \le -\gamma V, \tag{9}$$

where $\gamma > 0$ is a constant obtained from CARE. Satisfying (9) implies exponential convergence.

We can impose stronger bounds on convergence by constructing a *rapidly exponentially stable control Lyapunov function (RES-CLF)* that can be used to stabilize the output dynamics in a rapidly exponentially fashion (see Ames et al. (2014) for more details). Choosing $\varepsilon > 0$:

$$V_\varepsilon(\eta) := \eta^T\begin{bmatrix} \frac{1}{\varepsilon}I & 0 \\ 0 & I \end{bmatrix}P\begin{bmatrix} \frac{1}{\varepsilon}I & 0 \\ 0 & I \end{bmatrix}\eta =: \eta^TP_\varepsilon\eta.$$

It can be verified that this is a RES-CLF in Ames et al. (2014). Besides, the bounds on RES-CLF can be given as:

$$\alpha_1||\eta||^2 \le V_\varepsilon(\eta) \le \frac{\alpha_2}{\varepsilon^2}||\eta||^2, \tag{10}$$

where $\alpha_1, \alpha_2 > 0$ are the minimum and maximum eigenvalues of P, respectively. Differentiating (10) yields:

$$\dot{V}_\varepsilon(\eta) = L_FV_\varepsilon(\eta) + L_GV_\varepsilon(\eta)\mu, \tag{11}$$

where $L_FV_\varepsilon(\eta) = \eta^T(F^TP_\varepsilon + P_\varepsilon F)\eta$, $L_GV_\varepsilon(\eta) = 2\eta^TP_\varepsilon G$.

We can define a minimum norm controller which minimizes $\mu^T\mu$ subject to the inequality constraint:

$$L_FV_\varepsilon(\eta) + L_GV_\varepsilon(\eta)\mu \le -\frac{\gamma}{\varepsilon}V_\varepsilon(\eta), \tag{12}$$

which when satisfied implies exponential convergence. Therefore, we can define a class of controllers K_ε:

$$K_\varepsilon(\eta) = \{u \in \mathbb{R}^k : L_FV_\varepsilon(\eta) + L_GV_\varepsilon(\eta)u + \frac{\gamma}{\varepsilon}V_\varepsilon(\eta) \le 0\},$$

which yields the set of control values that satisfies the desired convergence rate.

3. UNMODELED DYNAMICS

Since the parameters are not perfectly known, the equation of motion, (1), computed with the given set of parameters will henceforth have ^ over the symbols. Therefore, D_a, C_a, G_a represent the actual model of the robot, and $\hat{D}$, $\hat{C}$, $\hat{G}$ represent the assumed model of the robot.

It is a well known fact that the inertial parameters of a robot are affine in the EOM (see Spong et al. (2006)). Therefore (1) can be restated as:

$$\Upsilon(q,\dot{q},\ddot{q})\Theta = B\mathrm{T}, \tag{13}$$

where $\Upsilon(q,\dot{q},\ddot{q})$ is the regressor Spong et al. (2006), and Θ is the set of base inertial parameters. Accordingly, Θ_a and $\hat{\Theta}$ are the actual and the assumed set of base inertial parameters respectively.

Computed Torque Redefined. The method of computed torque becomes very convenient to apply if the regressor and the inertial parameters are being computed simultaneously. If $\ddot{q}_d$ is the desired acceleration for the robot, the method of computed torque can be defined as:

$$B\mathrm{T}_{ct} = \Upsilon(q,\dot{q},\ddot{q}_d)\hat{\Theta}. \tag{14}$$

For convenience, the mapping matrix B on the left hand side of (14) will be omitted i.e., $B\mathrm{T}_{ct} = \mathrm{T}_{ct}$. Due to the difference in parameters, the dynamics will deviate from the nominal model, which is shown below:

Lemma 1. *Define:*

$$\Phi = \hat{D}^{-1}\Upsilon(q,\dot{q},\ddot{q}), \tag{15}$$

which is a function of the estimated parameters, $\hat{\Theta}$. If the control law used is (4) *combined with the computed torque* (14)*, then the resulting dynamics of the robot evolve as:*

$$\ddot{y} = \mu + J\Phi(\hat{\Theta} - \Theta_a). \tag{16}$$

Described in terms of η, we have the following:

$$\dot{\eta} = F\eta + G\mu + GJ\Phi\tilde{\Theta}, \;\; \dot{z} = \Psi(\eta, z), \tag{17}$$

where $\tilde{\Theta} = \hat{\Theta} - \Theta_a$. If $\tilde{\Theta} = 0$, we could apply $\mu(\eta) \in K_\varepsilon(\eta)$ to drive $\eta \to 0$. But since the parameters are uncertain, i.e., $\tilde{\Theta} \neq 0$, the resulting dynamics will be observed in the derivative of the Lyapunov function, V_ε, via

$$\dot{V}_\varepsilon(\eta,\mu) = \eta^T(F^T P_\varepsilon + P_\varepsilon F)\eta + 2\eta^T P_\varepsilon G\mu + 2\eta^T P_\varepsilon GJ\Phi\tilde{\Theta}, \tag{18}$$

where $\dot{\eta}$ is obtained via (17). The next section will establish the relationship between parameter uncertainty and the uncertain dynamics appearing in the CLF.

4. PARAMETER SENSITIVITY MEASURE

Due to the unmodeled dynamics, applying the controller, $\mu(\eta) \in K_\varepsilon(\eta)$, does not result in exponential convergence of the controller. The controller will still yield Global Uniform Ultimate Boundedness (GUUB) based on how the unmodeled dynamics affect $\dot{V}_\varepsilon$. The *parameter sensitivity measure*, ν, that quantifies the ultimate bound on the Lyapunov function V_ε is defined as:

$$\nu := \Upsilon(q,\dot{q},\ddot{q})\tilde{\Theta}. \tag{19}$$

It can be observed that: $\Upsilon(q,\dot{q},\ddot{q})\tilde{\Theta} = \Upsilon(q,\dot{q},\ddot{q})\hat{\Theta} - \Upsilon(q,\dot{q},\ddot{q})\Theta_a$, which is the difference between the actual and the expected torque being applied on the robot. Therefore, the *parameter sensitivity measure* is effectively the difference in torques applied on the robot.

Bounds on the Measure through RES-CLF. By (15) and (19), we have: $\hat{D}^{-1}\nu = \Phi\tilde{\Theta}$. Therefore, (18) can be expressed as:

$$\dot{V}_\varepsilon(\eta,\mu) = \eta^T(F^T P_\varepsilon + P_\varepsilon F)\eta + 2\eta^T P_\varepsilon G\mu + 2\eta^T P_\varepsilon GJ\hat{D}^{-1}\nu, \tag{20}$$

which is now a function of ν. This provides an important connection with Lyapunov theory, and the notion of parameter sensitivity is motivated by this observation. In other words, if the path of least parameter sensitivity is followed, then the convergence of the Lyapunov function to a value very close to zero can be realized. Therefore by (20), the control input μ must be chosen such that ν is well within the bounds specified. If a suitable controller is applied: $\mu(\eta) \in K_\varepsilon(\eta)$, the stability of the Lyapunov function can be achieved as long as the following equation is satisfied:

$$\dot{V}_\varepsilon \leq -\frac{\gamma}{\varepsilon}V_\varepsilon + 2\eta^T P_\varepsilon GJ\hat{D}^{-1}\nu \leq 0, \tag{21}$$

Since the measure ν is a function of the control input μ, (21) has an algebraic loop. But, given the control input, it is possible to restrict the outputs η within a certain region. Therefore, we first assume the bounds on the inertia matrix as:

$$\alpha_3 \leq ||D|| \leq \alpha_4, \;\; \hat{\alpha}_3 \leq ||\hat{D}|| \leq \hat{\alpha}_4, \tag{22}$$

where $\alpha_3, \alpha_4, \hat{\alpha}_3, \hat{\alpha}_4$ are constants (see Mulero-Martinez (2007); From et al. (2010)). Since the outputs are degree one functions of q, the jacobian is bounded by the constant: $||J|| \leq \kappa$.

By varying γ, it can be shown that the controller exponentially drives the outputs to a ball of radius β. In particular, by considering $\gamma_1 > 0, \gamma_2 > 0$, which satisfy $\gamma = \gamma_1 + \gamma_2$, we can rewrite $\frac{\gamma}{\varepsilon}V_\varepsilon = \frac{\gamma_1}{\varepsilon}V_\varepsilon + \frac{\gamma_2}{\varepsilon}V_\varepsilon$ in (21). The first term can thus be used to cancel the uncertain dynamics and yield exponential convergence until $||\eta||$ becomes sufficiently small. In other words, the outputs exponentially converge to an ultimate bound β as given by the following lemma.

Lemma 2. *Given the controllers $\mu(\eta) \in K_\varepsilon(\eta)$, $\bar{\mu}(\eta) \in \bar{K}_{\bar{\varepsilon}}(\eta)$, and $\delta > 0$, $\exists\ \beta > 0$ such that whenever $||\nu|| \leq \delta$, $\dot{V}_\varepsilon(\eta) < -\frac{\gamma_2}{\varepsilon}V_\varepsilon(\eta)\ \forall\ V_\varepsilon(\eta) > \beta$.*

The ultimate bound is given by $\beta = \frac{4\alpha_2^2\kappa^2\delta^2}{\alpha_1\hat{\alpha}_3^2\gamma_1^2\varepsilon^2}$. A discussion in this regard can be found in Dixon et al. (2004); Abdallah et al. (1991), where β is considered the uniform ultimate bound on the given controller.

It must be noted that Lemma 2 yields a low convergence rate γ_2 which is less than the original rate γ. The uncertain dynamics can be nullified separately by considering an auxiliary input $\bar{\mu}$ satisfying:

$$B\mathrm{T}_{ctn} = \hat{D}(q)\ddot{q}_d + \hat{C}(q,\dot{q})\dot{q} + \hat{G}(q) + B\bar{\mu}. \tag{23}$$

Note that this is not unique and other types of controllers can also be used. Computed torque with linear inputs appended have also been used in Slotine and Li (1987) in order to realize asymptotic convergence. The resulting dynamics of the outputs then reduces to:

$$\ddot{y} = \mu + J\Phi\tilde{\Theta} + J\hat{D}^{-1}B\bar{\mu}. \quad (24)$$

Therefore, $\dot{V}_\varepsilon$ for the new input can be reformulated as:

$$\dot{V}_\varepsilon(\eta, \mu, \bar{\mu}) = \eta^T(F^T P_\varepsilon + P_\varepsilon F)\eta + 2\eta^T P_\varepsilon G\mu + 2\eta^T P_\varepsilon GJ\hat{D}^{-1}(\nu + B\bar{\mu}). \quad (25)$$

Consider the input $\bar{\mu} = -\frac{1}{\bar{\varepsilon}}\Gamma^T G^T P_\varepsilon \eta$, where $\Gamma = J\hat{D}^{-1}B$. μ and $\bar{\mu}$ together form the computed torque+PD control on the robot. The end result is a positive semidefinite expression: $\frac{1}{\bar{\varepsilon}}\eta^T P_\varepsilon G\Gamma\Gamma^T G^T P_\varepsilon \eta \geq 0$, which motives the construction of a positive semidefinite function:

$$\bar{V}_\varepsilon(\eta) = \eta^T P_\varepsilon G\Gamma\Gamma^T G^T P_\varepsilon \eta =: \eta^T \bar{P}_\varepsilon \eta. \quad (26)$$

Using the property of positive semidefiniteness, we can establish new bounds on the outputs. Let $\mathcal{N}(\bar{P}_\varepsilon)$ be the null space of the matrix $\bar{P}_\varepsilon$. If $\eta \in \mathcal{N}(\bar{P}_\varepsilon)$, then $\bar{V}_\varepsilon(\eta) = 0$. Otherwise, for some $\alpha_7, \alpha_8 > 0$:

$$\alpha_7 ||\eta||^2 \leq \bar{V}_\varepsilon(\eta) \leq \frac{\alpha_8}{\varepsilon^4}||\eta||^2. \quad (27)$$

Note that (27) can be used to restrict the uncertain dynamics in (25). Utilizing these constructions, we can define the following class of controllers:

$$\bar{K}_{\bar{\varepsilon}}(\eta) = \{u \in \mathbb{R}^k : 2\eta^T P_\varepsilon GJ\hat{D}^{-1}Bu + \frac{1}{\bar{\varepsilon}}\bar{V}_\varepsilon(\eta) \leq 0\},$$

Lemma 2 can now be redefined to obtain the new ultimate bound β_η for the new control input (23).

Lemma 3. *Given the controllers $\mu(\eta) \in K_\varepsilon(\eta)$, $\bar{\mu}(\eta) \in \bar{K}_{\bar{\varepsilon}}(\eta)$, and $\delta > 0$, $\exists$ $\beta_\eta > 0$ such that whenever $||\nu|| \leq \delta$, $\dot{V}_\varepsilon(\eta) < -\frac{\gamma}{\varepsilon}V_\varepsilon(\eta)$ $\forall$ $V_\varepsilon(\eta) > \beta_\eta$.*

The resulting ultimate bound is: $\beta_\eta = \frac{4\alpha_2^2\kappa^2\delta^2\bar{\varepsilon}^2}{\alpha_7\alpha_9^2\hat{\alpha}_3^2\varepsilon^4}$, where $\alpha_9 = \frac{\bar{V}_\varepsilon}{V_\varepsilon}$. It can be inferred that:

$$V_\varepsilon(\eta(t)) \leq e^{-\frac{\gamma}{\varepsilon}t}V_\varepsilon(\eta(0)) \quad \text{for} \quad V_\varepsilon(\eta(0)) > \beta_\eta, \quad (28)$$

$$\text{or} \quad ||\eta(t)|| \leq \frac{1}{\varepsilon}\sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\frac{\gamma}{2\varepsilon}t}||\eta(0)|| \quad \text{for} \quad V_\varepsilon(\eta(0)) > \beta_\eta.$$

It must be noted that both $\varepsilon, \bar{\varepsilon}$ affect $||\nu||$. We can now consider utlizing the boundedness properties in underactuated systems given that the zero dynamics of the robot has a locally exponentially stable periodic orbit.

Zero Dynamics. Assume that there is an exponentially stable periodic orbit in the zero dynamics, denoted by $\mathcal{O}_z \subset Z$. This means that there is a Lyapunov function $V_z : Z \to \mathbb{R}_{\geq 0}$ such that in a neighborhood $B_r(\mathcal{O}_z)$ of $\mathcal{O}_z$ (see Hauser and Chung (1994)) it is exponentially stable. If $\mathcal{O} = \iota(\mathcal{O}_z) \subset X \times Z$ is the periodic orbit of the full order dynamics through the canonical embedding $\iota : Z \to X \times Z$, then by defining the composite Lyapunov function:

$$V_c(\eta, z) = \sigma V_\varepsilon(\eta) + V_z(z), \quad (29)$$

we can introduce a theorem that shows ultimate boundedness of the entire dynamics of the robot to the periodic orbit $\mathcal{O}$ under parameter uncertainty.

Theorem 1. *Given the controllers $\mu(\eta) \in K_\varepsilon(\eta)$, $\bar{\mu}(\eta) \in \bar{K}_{\bar{\varepsilon}}(\eta)$, and $\delta > 0$, $\exists$ $\beta_\eta, \beta_z > 0$ such that whenever $||\nu|| \leq \delta$, $V_c(\eta, z)$ is exponentially convergent $\forall$ $V_c(\eta, z) > \beta_\eta + \beta_z$.*

Proof of Theorem 1 is omitted due to space constraints but is an extension of Theorem 1 of Ames et al. (2014) which shows that by choosing a suitable σ, exponential convergence of V_c until the bound $\beta_\eta + \beta_z$ can be achieved.

5. HYBRID DYNAMICS

We now extend Theorem 1 to hybrid robotic systems which involve alternating phases of continuous and discrete dynamics. A hybrid system with a single continuous and a discrete event is defined as follows:

$$\mathscr{H} = \begin{cases} \dot{\eta} &= F\eta + G\mu + GJ\Phi\tilde{\Theta}, \\ \dot{z} &= \Psi(\Theta; \eta, z), \qquad \text{if } (\eta, z) \in \mathbb{D}\backslash\mathbb{S} \\ \eta^+ &= \Delta_\eta(\Theta, \eta^-, z^-), \\ z^+ &= \Delta_z(\Theta, \eta^-, z^-), \quad \text{if } (\eta^-, z^-) \in \mathbb{S} \end{cases} \quad (30)$$

It must be noted that the parameter vector Θ is included in zero dynamics, z, in (30), which is a reformulation of (7). $\mathbb{D}, \mathbb{S}$ are the domain and switching surfaces and are given by:

$$\mathbb{D} = \{(\eta, z) \in X \times Z : h(\eta, z) \geq 0\}, \quad (31)$$
$$\mathbb{S} = \{(\eta, z) \in X \times Z : h(\eta, z) = 0 \text{ and } \dot{h}(\eta, z) < 0\},$$

for some continuously differentiable function $h : X \times Z \to \mathbb{R}$. $\Delta(\Theta, \eta^-, z^-) = (\Delta_\eta(\Theta, \eta^-, z^-), \Delta_z(\Theta, \eta^-, z^-))$ is the reset map representing the discrete dynamics of the system. For the bipedal robot, AMBER, h represents the non-stance foot height and Δ represents the impact dynamics of the system. Plastic impacts are assumed. For $(q^-, \dot{q}^-) \in \mathbb{S}$, being the pre-impact angles and velocities of the robot, the post impact velocity for the assumed model $\dot{\hat{q}}^+$, and for the actual model $\dot{q}^+$ will be:

$$\dot{\hat{q}}^+ = (I - \hat{D}^{-1}\mathcal{J}^T(\mathcal{J}\hat{D}^{-1}\mathcal{J}^T)^{-1}\mathcal{J})\dot{q}^-,$$
$$\dot{q}^+ = D_a^{-1}(\hat{D} - \mathcal{J}^T R\tilde{D})\dot{\hat{q}}^+ - (I - \mathcal{J}^T R)\tilde{D}\dot{q}^-, \quad (32)$$

where $\mathcal{J}$ is the jacobian of the end effector where the impulse forces from the ground are acting on the robot, and $R = (\mathcal{J}D_a^{-1}\mathcal{J}^T)^{-1}\mathcal{J}D_a^{-1}$. Given the hybrid system (30), denote the flow as $\varphi_t(\Theta; \Delta(\Theta, \eta^-, z^-))$ with the initial condition $(\eta^-, z^-) \in \mathbb{S} \cap Z$.

Impact Measure. Using the impact model, measuring uncertainty of post-impact dynamics can be achieved by introducing an *impact measure*, ν_s, defined as follows:

$$\nu_s := \tilde{D}(q)\dot{q}^-, \quad (33)$$

It should be noted that the impact equations are Lipschitz continuous w.r.t. the impact measure ν_s. Accordingly, we have the following bounds on the impact map:

$$\begin{aligned} ||\Delta_\eta(\Theta_a, \eta^-, z^-) - \Delta_\eta(\hat{\Theta}, 0, z^-)|| \\ \leq ||\Delta_\eta(\Theta_a, \eta^-, z^-) - \Delta_\eta(\hat{\Theta}, \eta^-, z^-) \\ + \Delta_\eta(\hat{\Theta}, \eta^-, z^-) - \Delta_\eta(\hat{\Theta}, 0, z^-)|| \\ \leq L_1||\nu_s|| + L_2||\eta^-||, \end{aligned} \quad (34)$$

where L_1, L_2 are Lipschitz constants for Δ_η. Similarly:

$$||\Delta_z(\Theta_a, \eta^-, z^-) - \Delta_z(\hat{\Theta}, 0, z^-)|| \leq L_3||\nu_s|| + L_4||\eta^-||,$$

where L_3, L_4 are Lipschitz constants for Δ_z. In order to obtain bounds on the output dynamics for hybrid periodic orbits, it is assumed that $\mathscr{H}$ has a hybrid zero dynamics for the *assumed* model, $\hat{\Theta}$, of the robot. The hybrid zero dynamics can be described as:

$$\mathscr{H}|_Z = \begin{cases} \dot{z} &= \Psi(\hat{\Theta}; 0, z) \quad \text{if } z \in Z\backslash(\mathbb{S} \cap Z) \\ z^+ &= \Delta_Z(\hat{\Theta}, 0, z^-) \text{ if } z^- \in (\mathbb{S} \cap Z) \end{cases} \quad (35)$$

More specifically we assume that $\Delta_\eta(\hat{\Theta}; 0, z^-) = 0$, so that the surface Z is invariant under the discrete dynamics. Given (35), we can define the flow as $\varphi_t^z(\hat{\Theta}; \Delta(\hat{\Theta}, 0, z^-))$

with the initial state $(0, z^-) \in \mathbb{S} \cap Z$. If a periodic orbit $\mathcal{O}_z$ exists in (35), then there exists a periodic flow $\varphi_t^z(\hat{\Theta}; \Delta(\hat{\Theta}, 0, z^*))$ of period T^* for the fixed point $(0, z^*)$. Through the canonical embedding, the corresponding periodic flow of the periodic orbit $\mathcal{O}$ in (30) will be $\varphi_t(\hat{\Theta}; \Delta(\hat{\Theta}, 0, z^*))$. Note that existence of periodic orbit for the assumed model $\hat{\Theta}$ does not guarantee existence of the periodic orbit for the actual model Θ_a. Therefore, we define the Poincaré map $\mathcal{P} : \mathbb{S} \to \mathbb{S}$ given by:

$$\mathcal{P}(\Theta; \eta, z) = \varphi_{T(\Theta,\eta,z)}(\Theta; \Delta(\Theta, \eta, z)), \tag{36}$$

where Θ can be either Θ_a or $\hat{\Theta}$, and T is the time to impact function defined by:

$$T(\Theta; \eta, z) = \inf\{t \geq 0 : \varphi_t(\Theta; \Delta(\Theta, \eta, z)) \in \mathbb{S}\}. \tag{37}$$

It is shown in Ames et al. (2014) that T is Lipschitz continuous. Therefore, for constants $\alpha_{15} < 1 < \alpha_{16}$, $\alpha_{15} T^* \leq T(\Theta, \eta, z) \leq \alpha_{16} T^*$. These constants depend on the deviation from the nominal model $\hat{\Theta}$ and how far (η, z) is from the fixed point. The corresponding Poincaré map $\rho : \mathbb{S} \cap Z \to \mathbb{S} \cap Z$ for the hybrid zero dynamics (35) is termed the restricted Poincaré map:

$$\rho(z) = \varphi^z_{T_\rho(z)}(\hat{\Theta}; \Delta(\hat{\Theta}, 0, z)), \tag{38}$$

where T_ρ is the restricted time to impact function which is given by $T_\rho(z) = T(\hat{\Theta}; 0, z)$. Without loss of generality, we can assume that $\hat{\Theta} = 0$, $z^* = 0$. The following Lemma will introduce the relationship between time to impact, Poincaré functions with η, ν_s.

Lemma 4. *Let $\mathcal{O}_z$ be the periodic orbit of the hybrid zero dynamics $\mathscr{H}|_Z$ transverse to $\mathbb{S} \cap Z$ for the nominal model $\hat{\Theta}$. Given the controllers $\mu(\eta) \in K_\varepsilon(\eta)$, $\bar{\mu}(\eta) \in \bar{K}_{\bar{\varepsilon}}(\eta)$ for the actual model Θ_a which render ultimate boundedness on $V_\varepsilon(\eta)$, $(\eta, z) \in \mathbb{S} \cap Z$ such that $V_\varepsilon(\Delta_\eta(\Theta, \eta, z)) > \beta_\eta$ for the given robot model Θ, and $r > 0$ such that $(\eta, z) \in B_r(0, 0)$, there exist finite constants $A_1, A_2, A_3, A_4 > 0$ such that:*

$$||T(\Theta_a; \eta, z) - T_\rho(z)|| \leq A_1||\eta|| + A_2||\nu_s|| \tag{39}$$

$$||\mathcal{P}(\Theta_a; \eta, z) - \rho(z)|| \leq A_3||\eta|| + A_4||\nu_s|| \tag{40}$$

Proof (Sketch). Let $\mu_1 \in \mathbb{R}^{2(n-k)}$, $\mu_2 \in \mathbb{R}^{2k}$ be constant vectors. Define an auxiliary time to impact function: $T_B(\mu_1, \mu_2, z) = \inf\{t \geq 0 : h(\mu_1, \varphi_t^z(0, \Delta(0, 0, z)) + \mu_2) = 0\}$, which is Lipschitz continuous with the Lipschitz constant L_B: $||T_B(\mu_1, \mu_2, z) - T_\rho(z)|| \leq L_B(||\mu_1|| + ||\mu_2||)$.

Let $(\eta_1(t), z_1(t))$ satisfy $\dot{z}_1 = \Psi(\Theta_a; \eta_1(t), z_1(t))$ with $\eta_1(0) = \Delta_\eta(\Theta_a, \eta, z)$ and $z_1(0) = \Delta_z(\Theta_a, \eta, z)$. Similarly let $z_2(t)$ satisfy $\dot{z}_2(t) = \Psi(0; 0, z_2(t))$ such that $z_2(0) = \Delta_z(0, 0, z)$. The bounds on the $||\eta||$ can now be given as $||\eta_1(0)|| = ||\Delta_\eta(\Theta, \eta, z) - \Delta_\eta(0, 0, z)|| \leq L_1||\nu_s|| + L_2||\eta||$, which is obtained through (34). Since $V_\varepsilon(\eta_1) > \beta_\eta$, we use (59) of Lemma 1 of Ames et al. (2014) and (28) to obtain $||\mu_1||$. Similartly, to obtain $||\mu_2||$, we use the Gronwall-Bellman argument from the proof of Lemma 1 of Ames et al. (2014) to get the following inequality:

$$||z_1(t) - z_2(t)|| \leq (C_2||\eta|| + C_3||\nu_s||)\, e^{L_q t}, \tag{41}$$

where C_2, C_3, L_q are constants. Proof of (39) can now be obtained by substituting for $||\mu_1||, ||\mu_2||$.

Similarly, the derivation of (54) of Lemma 1 of Ames et al. (2014) is used to obtain (40). □

Main Theorem. We can now introduce the main theorem of the paper. Similar to the continuous dynamics, it is assumed that the periodic orbit $\mathcal{O}_Z$ is exponentially stable in the hybrid zero dynamics.

Theorem 2. *Let $\mathcal{O}_z$ be an exponentially stable periodic orbit of the hybrid zero dynamics $\mathscr{H}|_Z$ transverse to $S \cap Z$ for the nominal model $\hat{\Theta}$. Given the controllers $\mu(\eta) \in K_\varepsilon(\eta)$, $\bar{\mu}(\eta) \in \bar{K}_{\bar{\varepsilon}}(\eta)$ for the hybrid system $\mathscr{H}$. Given $r > 0$ such that $(\eta, z) \in B_r(0, 0)$, there exist $\delta > 0, \delta_s > 0$ such that whenever $||\nu|| < \delta$, $||\nu_s|| < \delta_s$, the orbit $\mathcal{O} = \iota(\mathcal{O}_Z)$ is uniformly ultimately bounded by $\beta_\eta + \beta_z$.*

Proof (Sketch). Results of Lemma 4 and the exponential stability of $\mathcal{O}_Z$ imply that there exists $r > 0$ such that $\rho : B_r(0) \cap (S \cap Z) \to B_r(0) \cap (S \cap Z)$ is well defined for all $z \in B_r(0) \cap (S \cap Z)$ and $z_{k+1} = \rho(z_k)$ is locally exponentially stable, i.e., $||z_k|| \leq N\xi^k||z_0||$ for some $N > 0$, $0 < \xi < 1$ and all $k \geq 0$. Therefore, by the converse Lyapunov theorem for discrete systems, there exists an exponentially convergent Lyapunov function V_ρ, defined on $B_r(0) \cap (S \cap Z)$ for some $r > 0$ (possibly smaller than the previously defined r). It must be first ensured that the region β_η must be within the bounds defined by r. For $||\eta|| = r$, $\beta_\eta < V_\varepsilon(\eta)$ is ensured through the following condition: $\beta_\eta < V_\varepsilon(\eta) < \frac{\alpha_2}{\varepsilon^2}||\eta||^2 < \frac{\alpha_2}{\varepsilon^2} r^2$. From Lemma 2 we have: $\frac{4\alpha_2^3\kappa^2\delta^2}{\alpha_1^2\alpha_3^2\gamma_1^2\varepsilon^4} < \frac{\alpha_2}{\varepsilon^2} r^2 \implies \delta < \frac{\alpha_1\alpha_3\gamma_1\varepsilon^2}{2\alpha_2\kappa} r$.

For the RES-CLF V_ε, denote its restriction to the switching surface by $V_{\varepsilon,\eta} = V_\varepsilon|_S$, which is exponentially convergent. With these two Lyapunov functions we define the following candidate Lyapunov function on $B_r(0, 0) \cap S$:

$$V_P(\eta, z) = V_\rho(z) + \sigma V_{\varepsilon,\eta}(\eta) \tag{42}$$

The idea is to show that there exists a bounded region β into which the dynamics of the robot exponentially converge. Since the origin is an exponentially stable equilibrium for $z_{k+1} = \rho(z_k)$, we have the following inequalities:

$$\begin{aligned} ||\mathcal{P}_z(\Theta_a; \eta, z)|| &= ||\mathcal{P}_z(\Theta_a; \eta, z) - \rho(z) + \rho(z) - \rho(0)|| \\ &\leq A_3||\eta|| + A_4||\nu_s|| + L_\rho||z|| \\ ||\rho(z)|| &\leq N\xi||z||, \end{aligned} \tag{43}$$

where L_ρ is the Lipschitz constant for ρ. Therefore:

$$\begin{aligned} &V_\rho(\mathcal{P}_z(\Theta_a; \eta, z)) - V_\rho(\rho(z)) \\ &\quad \leq \alpha_{20}(A_3||\eta|| + A_4||\nu_s||) \\ &\qquad\qquad (A_3||\eta|| + A_4||\nu_s|| + (L_\rho + N\xi)||z||), \end{aligned} \tag{44}$$

where α_{20} is equivalent to the constant r_4 given in (61) of proof of Theorem 2 in Ames et al. (2014). It follows that:

$$\begin{aligned} V_\rho(\mathcal{P}_z(\Theta_a, \eta, z)) - V_\rho(z) = \;& V_\rho(\mathcal{P}_z(\Theta_a; \eta, z)) - V_\rho(\rho(z)) \\ & + V_\rho(\rho(z)) - V_\rho(z). \end{aligned}$$

Combining the entire Lyapunov function we have:

$$V_P(\mathcal{P}(\Theta_a; \eta, z)) - V_P(\eta, z) \leq - \begin{bmatrix} ||\eta|| \\ ||z|| \\ ||\nu_s|| \end{bmatrix}^T \Lambda_{\mathscr{H}} \begin{bmatrix} ||\eta|| \\ ||z|| \\ ||\nu_s|| \end{bmatrix}$$

where the symmetric matrix $\Lambda_{\mathscr{H}} \in \mathbb{R}^{3\times 3}$, is obtained by collecting the constant terms. It can found in the derivation of Theorem 2 of Ames et al. (2014) that by choosing a suitable σ positive definiteness of $\Lambda_{\mathscr{H}}$ can be established to render the discrete Lyapunov function V_P exponentially convergent. The upper bound on the impact measure is obtained by applying the contraint on the above equation to ensure exponential convergence. □

6. SIMULATION RESULTS AND CONCLUSIONS

In this section, we will invesigate how the uncertainty in parameters affects the stability of the controller applied to the 5-DOF bipedal robot AMBER shown in Fig. 1. The model, Θ_a, which has 61 parameters is picked such that the error is 30% compared to the assumed model $\hat{\Theta}$.

To realize walking on the robot, the actual and desired outputs are chosen as in Yadukumar et al. (2012) (specifically, see (6) for determining the actual and the desired outputs). The end result is outputs of the form $y(q) = y_a(q) - y_d(q)$ which must be driven to zero. Therefore, the objective of the computed torque controller (5) combined with K_ε for μ and $\bar{K}_{\bar{\varepsilon}}$ for $\bar{\mu}$ is to drive $y \to 0$. For the nominal model $\hat{\Theta}$ a stable walking gait is observed. In other words, a stable hybrid periodic orbit is observed for the assumed given model. Since, the actual model of the robot has an error of 30%, applying the controller yields the dynamics that evolves as shown in (24). The value of ε chosen was 1, and $\bar{\varepsilon}$ was 2. Fig. 2 shows the comparison between actual and desired outputs, and Fig. 3 shows the Lyapunov function V_ε. It can be observed that after every impact, V_ε is thrown outside the ball defined by $\beta_\eta \approx 0.04$ and the controllers act to get back into β_η before the next impact.

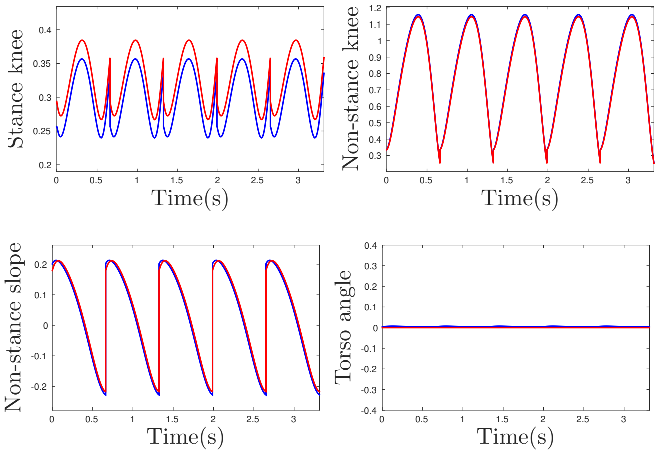

Fig. 2. Actual (blue) and desired (red) outputs as a function of time are shown here.

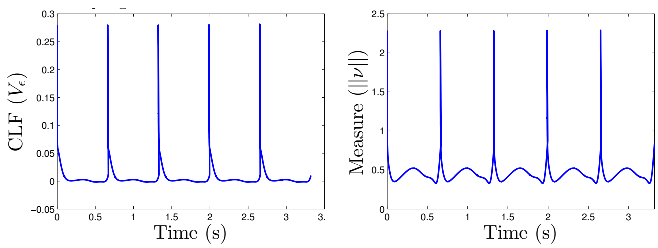

Fig. 3. The RES-CLF (left) and the measure (right) as a function of time are shown here. $\dot{V}_\varepsilon$ crosses 0 in every step, but the CLF is still seen to be ultimately bounded by β_η.

Conclusions. The concept of a measure for evaluating the robustness to parameter uncertainty of hybrid system models of robots was introduced. This relationship is created by introducing the formula for the parameter sensitivity measure. The notion of measure is extended to hybrid systems (which includes discrete events) by considering the impact measure. This impact measure along with the sensitivity measure determine the output boundedness of the composite Lyapunov function for hybrid periodic orbits of the robot. This was then verified by realizing a stable walking gait on AMBER having a parameter error of 30%. It is important to observe that while the parameter sensitivity measure yields the difference between the actual and the predicted torque applied on the robot, the impact measure yields the impulsive ground reaction forces acting upon the robot during impacts. Future work will be devoted to evaluating the formal results of this paper experimentally.

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