Blowing Up Affine Hybrid Systems

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***Abstract*— In this paper we construct the "blow up" of an affine hybrid system H, i.e., a new affine hybrid system $\mathrm{Bl}(\mathbf{H})$ in which H is embedded, that does not exhibit Zeno behavior. We show the existence of a bijection Υ between periodic orbits and equilibrium points of H and $\mathrm{Bl}(\mathbf{H})$ that preserves stability; we refer to this property as $\mathscr{P}$-stability equivalence.**

I. INTRODUCTION

If $\mathbf{H}$ is an affine hybrid system, we introduce its blow up $\mathrm{Bl}(\mathbf{H})$ which is also an affine hybrid system. The primary benefit of considering $\mathrm{Bl}(\mathbf{H})$ is that it is not Zeno, although its structure suggests many other interesting properties not generally found in affine hybrid systems. In order to demonstrate that $\mathrm{Bl}(\mathbf{H})$ is in some way equivalent to $\mathbf{H}$, $\mathscr{P}$-stability equivalence is introduced. If $\mathscr{O}^{\mathbf{H}}$ is the set of equilibrium points and periodic orbits of $\mathbf{H}$, then two affine hybrid systems $\mathbf{H}$ and $\mathbf{G}$ are $\mathscr{P}$-stability equivalent if there exists a bijection $\Upsilon : \mathscr{O}^{\mathbf{H}} \to \mathscr{O}^{\mathbf{G}}$ such that

$$\mu \in \mathscr{O}^{\mathbf{H}} \text{ is } \mathscr{P}\text{-stable} \quad \Leftrightarrow \quad \Upsilon(\mu) \in \mathscr{O}^{\mathbf{G}} \text{ is } \mathscr{P}\text{-stable}$$

where $\mathscr{P}$ is stability in the sense of Lyapunov, asymptotic stability or exponential stability. The purpose of this paper is to prove the following theorem:

Main Theorem: *The affine hybrid systems* $\mathbf{H}$ *and* $\mathrm{Bl}(\mathbf{H})$ *are* $\mathscr{P}$*-stability equivalent, and* $\mathrm{Bl}(\mathbf{H})$ *is not Zeno.*

The importance of the Main Theorem is that rather than attempting to determine whether an affine hybrid system is Zeno (which currently is not possible), analysis can be carried out on $\mathrm{Bl}(\mathbf{H})$ where there is no Zeno behavior. Additionally, most analysis on the stability of hybrid systems, or even switched systems, assumes that such systems are not Zeno, cf. [2], [5]-[9]. Because of the Main Theorem, this assumption automatically holds for $\mathrm{Bl}(\mathbf{H})$, and $\mathrm{Bl}(\mathbf{H})$ is $\mathscr{P}$-stability equivalent to $\mathbf{H}$, so the assumption is not restrictive. $\mathrm{Bl}(\mathbf{H})$ displays additional desirable properties that are not found in general affine hybrid systems. Its structure closely resembles a switched system, implying that $\mathrm{Bl}(\mathbf{H})$ might provide a way to apply the analysis carried out on switched systems to affine hybrid systems; since there are considerably more results for switched systems, this would be an important connection. In the future, these and other properties of $\mathrm{Bl}(\mathbf{H})$ will be investigated.

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The structure of this paper is as follows:

Section II: Reviews the definition of an affine hybrid system; more details and examples can be found in [1].

Section III: Begins by introducing hybrid executions. These are used to define the important types of equilibrium points and periodic orbits of an affine hybrid system and discuss their stability. This section concludes with the definition of $\mathscr{P}$-stability equivalence.

Section IV: $\mathrm{Bl}(\mathbf{H})$ is constructed.

Section V: Relationships between $\mathbf{H}$ and $\mathrm{Bl}(\mathbf{H})$ are discussed, the most important being: there is a bijection $\Upsilon : \mathscr{O}^{\mathbf{H}} \to \mathscr{O}^{\mathrm{Bl}(\mathbf{H})}$ that is explicitly computable.

Section VI: The Main Theorem is proven.

II. AFFINE HYBRID SYSTEMS

This section introduces the notion of an affine hybrid system. An affine hybrid system consists of the following data: a set of discrete states, domains, edges and vector fields. The discrete states provide a way to index the domains. The domains are affine sets, i.e., sets that are affinely constrained. The edges provide a relationship between two faces of two domains; each edge has a source which is the face of a domain and a target which is also the face of a domain. It is required that there exists an affine transformation between the source and the target of each edge; thus each edge gives rise to a transition map, which is an affine transformation, from the source of the edge to the target of the edge. The set of vector fields is a collection of vector fields that are globally Lipschitz.

2.1 (Discrete states): The set of *discrete states* is a finite set $Q = \{1, ..., m\}$.

2.2 (Domains): The set of *domains* is the set $D = \{D_i\}_{i \in Q}$, where each $D_i \subset \mathbb{R}^n$ is an n-dimensional affine set, i.e., a set that is affinely constrained. For each set D_i, there exists a matrix $A_i \in \mathbb{R}^{k_i \times n}$ and a vector $a_i \in \mathbb{R}^{k_i}$ such that

$$x \in D_i \qquad \Leftrightarrow \qquad A_i x + a_i \geq 0,$$

where k_i is the number of $n-1$ dimensional affine sets contained in the boundary of D_i; these are called the *faces* of D_i. The faces of D_i can be indexed by introducing the indexing set,

$$F_i = \{1, ..., k_i\}, \qquad i \in Q.$$

The j^{th} face of D_i is denoted by $\mathrm{Face}_j(D_i)$, where $j \in F_i$. We can pick an indexing of the faces of D_i by letting $\mathrm{Face}_j(D_i)$ be the affine set determined by the j^{th} row of A_i. More precisely, if $(A_i)_{j*}$ is the j^{th} row of A_i and $(a_i)_j$ is the j^{th} entry of a_i, then

$$\begin{array}{c} x \in \mathrm{Face}_j(D_i) \\ \Updownarrow \\ \left(\begin{array}{c} A_i \\ -(A_i)_{j*} \end{array} \right) x + \left(\begin{array}{c} a_i \\ -(a_i)_j \end{array} \right) \geq 0. \end{array} \tag{1}$$

This definition can be extended to affine sets D_i with $\dim(D_i) \leq n$ in the obvious manner.

2.3 (Edges): For a set U with $U = \prod_{i=1}^{n} U_i$, denote the projections on each of the factors of U by $\pi_i : U \to U_i$. Define the set of *edges* as a set

$$E \subseteq \{((i,j),(k,l))\}_{(i,j) \in Q \times Q, \ \ (k,l) \in F_i \times F_j},$$

satisfying the condition that for each $e \in E$, there exists a map $T_e(x) = R_e x + p_e$, with $(R_e, p_e) \in SE(n)$, such that

$$T_e(\mathrm{Face}_{\pi_3(e)}(D_{\pi_1(e)})) = \mathrm{Face}_{\pi_4(e)}(D_{\pi_2(e)}).$$

To simplify notation, write

$$\begin{aligned} \mathrm{Source}(e) &= \mathrm{Face}_{\pi_3(e)}(D_{\pi_1(e)}), \\ \mathrm{Target}(e) &= \mathrm{Face}_{\pi_4(e)}(D_{\pi_2(e)}). \end{aligned}$$

Given an edge $e \in E$, the affine transformation $T_e(x) = R_e x + p_e$ from $\mathrm{Source}(e)$ to $\mathrm{Target}(e)$ is called the *transition map*. The set of transition maps is the set $T = \{T_e\}_{e \in E}$.

2.4 (Vector fields): A set of vector fields is a set $V = \{V_i\}_{i \in Q}$, where V_i is a Lipschitz vector field on $\mathbb{R}^n$. The flow of V_i on D_i is denoted by $\varphi_i(t, x)$ for $x \in D_i$.

Definition 2.1: An affine hybrid system is a tuple

$$\mathbf{H} = (Q, D, E, V).$$

Note 2.1: From this point on, for the sake of brevity, we will refer to "affine hybrid systems" as "hybrid systems." When dealing with multiple hybrid systems, the superscripts are added to avoid confusion between the hybrid systems. For example, two hybrid systems $\mathbf{H}$ and $\mathbf{G}$ are given by the tuples $\mathbf{H} = (Q^{\mathbf{H}}, D^{\mathbf{H}}, E^{\mathbf{H}}, V^{\mathbf{H}})$ and $\mathbf{G} = (Q^{\mathbf{G}}, D^{\mathbf{G}}, E^{\mathbf{G}}, V^{\mathbf{G}})$.

2.5: If for some $e \in E$, $T_e(x) = x$, then we say that the transition map associated with the edge e is the identity map. This implies that $\mathrm{Source}(e) = \mathrm{Target}(e)$. Since these are affine sets, we can define a matrix A_e and vector a_e such that

$$A_e x + a_e \geq 0 \quad \Leftrightarrow \quad x \in \mathrm{Source}(e) = \mathrm{Target}(e)$$

In particular, these affine constraints could be the affine constraints determining $\mathrm{Source}(e)$ or $\mathrm{Target}(e)$ as given by Equation (1).

A very special class of hybrid systems is the class hybrid systems in which every transition map is the identity. This is the class of hybrid systems we will consider in this paper; hence we make the following assumption.

Assumption 2.1: For the hybrid system $\mathbf{H}$, every transition map is the identity.

This assumption is not as restrictive as one might think due to the main theorem of [1]: Every compact hybrid system is *spatially equivalent* to a hybrid system in which every transition map is the identity. Extending the results of this paper to arbitrary hybrid systems through the use of spatial equivalence will be the topic of future consideration.

III. From Executions to $\mathscr{P}$-Stability Equivalence

This section begins with the definition of a hybrid execution which varies somewhat from the standard definition (cf. [11],[12]). With this definition the hybrid flow can be defined; it is analogous to the flow of a dynamical system. Using this, the important types of equilibrium points and periodic orbits of hybrid systems are introduced, and the different forms of stability that these objects can display are discussed. This section culminates with the definition of $\mathscr{P}$-stability equivalence. Essentially, two hybrid systems $\mathbf{H}$ and $\mathbf{G}$ are $\mathscr{P}$-stability equivalent if they display the same qualitative behavior with respect to stability.

3.1 (Hybrid Execution): Let Λ be a finite or countably infinite indexing set such that if $N = |\Lambda| - 1$ then $\Lambda = \{0, 1, ..., N\}$ if N is finite, and $\Lambda = \mathbb{Z}^* = \{0, 1, ...\}$ if $N = \infty$. Also define $\Lambda_+ = \{1, ..., N\}$ if N is finite and $\Lambda_+ = \mathbb{Z}^+ = \{1, 2, ...\}$ if $N = \infty$.

A *hybrid time sequence* is a finite or infinite sequence of real numbers $\tau = \{\tau_i\}_{i \in \Lambda}$, with

$$0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_i \leq \cdots,$$

a *hybrid edge sequence* $\eta = \{\eta_i\}_{i \in \Lambda_+}$ is a sequence of edges $\eta_i \in E$, and a *sequence of initial conditions* is a sequence $\xi = \{\xi_i\}_{i \in \Lambda}$ with $\xi_i \in \mathbb{R}^n$.

A *hybrid execution* is a tuple $\chi = (\tau, \eta, \xi)$ satisfying the following conditions:

For all $0 \leq i < N$,

$$\begin{aligned} \tau_{i+1} &= \min\{t \geq \tau_i : \varphi_{\pi_1(\eta_{i+1})}(t - \tau_i, \xi_i) \in \partial D_{\pi_1(\eta_{i+1})}\} \\ \xi_{i+1} &= T_{\eta_{i+1}}(\varphi_{\pi_1(\eta_{i+1})}(\tau_{i+1} - \tau_i, \xi_i)) \in \mathrm{Target}(\eta_{i+1}), \end{aligned}$$

and $\pi_1(\eta_{i+1}) = \pi_2(\eta_i)$ for $1 \leq i < N$.

If N is finite, define an additional element τ_{N+1} as follows: if $\varphi_{\pi_2(\eta_N)}(t - \tau_N, \xi_N) \in \partial D_{\pi_2(\eta_N)}$ for some finite $t > \tau_N$, then define

$$\tau_{N+1} = \min\{t > \tau_N : \varphi_{\pi_2(\eta_N)}(t - \tau_N, \xi_N) \in \partial D_{\pi_2(\eta_N)}\},$$

otherwise set $\tau_{N+1} = \infty$. The element τ_{N+1} is the *termination time* of a finite execution.

3.2: Given a hybrid execution χ, the map $d : \Lambda \to Q$, defined by $d(i) = \pi_1(\eta_{i+1})$ if $0 \leq i < N$ and $d(N) = \pi_2(\eta_N)$ if N is finite, is the *discrete state evolution*. It sometimes is convenient to associate to an $\eta_i \in \eta$ the corresponding edge in E, therefore if $E = \{e_1, ..., e_k\}$, define a map $\rho : \Lambda_+ \to \{1, ..., k\}$ such that $\eta_i = e_{\rho(i)}$; this can be thought of as the *evolution of edges*.

3.3: An execution χ is called finite if N is finite and infinite if $N = \infty$. Let

$$\begin{aligned} S(\mathbf{H}) &= \{ \text{ Set of executions of } \mathbf{H}\} \\ S_0(\mathbf{H}) &= \{ \text{ Set of finite executions of } \mathbf{H}\} \\ S_\infty(\mathbf{H}) &= \{ \text{ Set of infinte executions of } \mathbf{H}\}. \end{aligned}$$

An execution χ is *Zeno* if $\chi \in S_\infty(\mathbf{H})$ and $\lim_{i\to\infty} \tau_i = \tau_\infty$, where τ_∞ is a finite real number. A hybrid system $\mathbf{H}$ is Zeno if there is at least one Zeno execution. There have been numerous attempts to determine which hybrid systems display Zeno behavior, cf. [4], [6], [12]. When multiple hybrid executions are being discussed (possibly of different hybrid systems), we will write d^χ,ρ^χ, N^χ, Λ^χ and $\mathcal{T}^\chi$ to remove ambiguity.

3.4: Since we are assuming that $T_e = \text{id}$ for every $e \in E$, given an execution $\chi = (\tau, \eta, \xi)$ we can define the hybrid flow of χ which is roughly analogous to the flow of a differential equation. Let

$$a_i^\chi(t) = \begin{cases} 1 \quad if \qquad t \in [\tau_i, \tau_{i+1}],\ 0 \leq i < N^\chi \\ 1 \quad if \quad t \in [\tau_N, \tau_{N+1}),\ i = N^\chi,\ \chi \in S_0(\mathbf{H}) \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

and set

$$\mathcal{T}^\chi = \{t \in \mathbb{R} : \ a_i^\chi(t) > 0, \text{ for some } i \in \Lambda^\chi\}.$$

Define the *hybrid flow* as

$$\varphi^\chi(t, \xi_0) = \frac{1}{\sum_{i\in\Lambda} a_i^\chi(t)} \sum_{i\in\Lambda} a_i^\chi(t) \varphi_{d^\chi(i)}(t - \tau_i, \xi_i),$$

for $t \in \mathcal{T}^\chi$. This implies that $\varphi(t, \xi_0) = \varphi_{d^\chi(i)}(t - \tau_i, \xi_i)$ when $t \in [\tau_i, \tau_{i+1}]$. Note that hybrid flows are defined uniquely by an execution.

3.5: Hybrid systems display more types of equilibrium points and periodic orbits than classical dynamical systems (cf. [9]). We will consider the following:

CSEP = *Continuous state equilibrium point*: A point x^* such that $V_i^{\mathbf{H}}(x^*) = 0$ for some $i \in Q$.

DSPO = *Discrete state periodic orbit*: A point x^* such that for every execution $\chi \in S_\infty(\mathbf{H})$ with $\xi_0 = x^*$, $\tau_i = 0$ for all $i \in \Lambda^\chi$ and

$$d(i) = d(i + pK^\chi) \qquad d(i) \neq d(j) \quad j \neq i + pK^\chi$$

for some integer $K^\chi > 0$ (dependent on χ) and all $p \in \mathbb{Z}^*$. Discrete state periodic orbits can imply the existence of Zeno executions.

CSPO = *Continuous state periodic orbit*: A set $\gamma \subset D_i^{\mathbf{H}}$ (not a point) that is a periodic orbit of $V_i^{\mathbf{H}}$, i.e., there exists a finite T such that for each $\xi_0 \in \gamma$, $\varphi_i(pT, \xi_0) = \xi_0$ for all $p \in \mathbb{Z}$.

MSPO = *Mixed state periodic orbit*: A connected set $\gamma \subset \mathbb{R}^n$ (not a point) such that

$$\gamma = \bigcup_{\substack{\chi \in S_\infty(\mathbf{H}) \\ \xi_0 \in \gamma}} \{\varphi^\chi(t, \xi_0) : t \in \mathbb{R}^+\},$$

where for every $\chi \in S_\infty(\mathbf{H})$ with $\xi_0 \in \gamma$, $\mathcal{T}^\chi = \mathbb{R}^+$ and

$$\begin{aligned} \varphi^\chi(t, \xi_0) &= \varphi^\chi(t + pT^\chi) \qquad \varphi^\chi(t) \in \gamma \quad \forall\ t \in \mathbb{R}^+ \\ d(i) &= d(i + pK^\chi) \qquad d(i) \neq d(j) \quad j \neq i + pK^\chi \end{aligned}$$

for a real number $T^\chi > 0$, an integer $K^\chi > 0$ (dependent on χ) and all $p \in \mathbb{Z}^*$.

It is useful to talk about the set of all equilibrium points and periodic orbits. Let

$$\mathscr{O}^{\mathbf{H}} = \left\{ \begin{array}{c} \textbf{CSEP}\text{'s , } \textbf{DSPO}\text{'s , } \textbf{CSPO}\text{'s} \\ \text{and } \textbf{MSPO}\text{'s of } \mathbf{H} \end{array} \right\}.$$

If $\mu \in \mathscr{O}^{\mathbf{H}}$ then μ is either a point (in which case it is a **CSEP** or a **DSPO**), or it is a set not equal to a point (in which case it is a **CSPO** or a **MSPO**).

3.6: Let $B_\delta(\mu)$ be a neighborhood of $\mu \in \mathscr{O}^{\mathbf{H}}$, i.e., for all $x \in B_\delta(\mu)$, $\|x - \mu\| = \min_{u\in\mu} \|x - u\| < \delta$. Consider the following forms of stability of μ:

For all $\chi \in S(\mathbf{H})$ with $\xi_0 \in B_\delta(\mu)$, $\mu \in \mathscr{O}^{\mathbf{H}}$ is

LYP = *Stable in the sense of Lyapunov*: If there exists an $\epsilon > 0$ such that for all $t \in \mathcal{T}^\chi$

$$\|\varphi^\chi(t, \xi_0) - \mu\| \leq \epsilon.$$

ASY = *Asymptotically stable*: If

$$\lim_{t\to\sup \mathcal{T}^\chi} \|\varphi^\chi(t, \xi_0) - \mu\| \to 0.$$

EXP = *Exponentially stable*: If there exists and $\alpha, M > 0$ such that for all $t \in \mathcal{T}^\chi$

$$\|\varphi^\chi(t, \xi_0) - \mu\| \leq Me^{-\alpha t}\|\xi_0 - \mu\|.$$

A stability property is denoted by $\mathscr{P}$ = **LYP**, **ASY**, or **EXP**.

Definition 3.1: Two hybrid systems $\mathbf{H}$ and $\mathbf{G}$ are $\mathscr{P}$-*stability equivalent* if there exists a bijection $\Upsilon : \mathscr{O}^{\mathbf{H}} \to \mathscr{O}^{\mathbf{G}}$ such that

$$\mu \text{ is } \mathscr{P}\text{-stable} \quad \Leftrightarrow \quad \Upsilon(\mu) \text{ is } \mathscr{P}\text{-stable}$$

IV. The Blow Up of a Hybrid System

In this section the blow up of a hybrid system is defined constructively. The underlying idea is simple and, as the name suggests, was originally motivated by the blow up of a singular variety in algebraic geometry (more specifically, it was originally motivated by the example on page 28 of Hartshorne's *Algebraic Geometry* [3]). Although a hybrid systems does not possess the same algebraic structure of an algebraic variety–and so cannot be blown up like an algebraic variety–the name "blow up" is given to this construction since the blow up of a hybrid system eliminates Zeno just as the blow up of a singular variety eliminates singularities.

4.1 (Construction of $\mathrm{Bl}(\mathbf{H})$***):*** The blow up of a hybrid system $\mathbf{H}$ is a hybrid system

$$\mathrm{Bl}(\mathbf{H}) = (Q^{\mathrm{Bl}(\mathbf{H})}, D^{\mathrm{Bl}(\mathbf{H})}, E^{\mathrm{Bl}(\mathbf{H})}, V^{\mathrm{Bl}(\mathbf{H})}),$$

where the individual elements are defined as follows:

$\underline{Q^{\mathrm{Bl}(\mathbf{H})}}$: If $Q^{\mathbf{H}} = \{1, ..., m\}$ and $k = |E^{\mathbf{H}}|$, then

$$Q^{\mathrm{Bl}(\mathbf{H})} = \{1, ..., m+k\}.$$

$\underline{D^{\mathrm{Bl}(\mathbf{H})}}$: Let $q \in \mathbb{R}^m$ and λ_i be the i^{th} standard basis vector of $\mathbb{R}^m$. If $i \leq m$, then $D_i^{\mathbf{H}}$ is determined by the affine constraints $A_i^{\mathbf{H}} x + a_i^{\mathbf{H}}$. Define $D_i^{\mathrm{Bl}(\mathbf{H})}$ to be the affine set given by the affine constraints

$$\begin{pmatrix} A_i^{\mathbf{H}} & 0 \\ 0 & I \\ 0 & -I \end{pmatrix} \begin{pmatrix} x \\ q \end{pmatrix} + \begin{pmatrix} a_i^{\mathbf{H}} \\ -\lambda_i \\ \lambda_i \end{pmatrix} \geq 0.$$

By indexing the elements of $E^{\mathbf{H}}$ such that $\{e_1^{\mathbf{H}}, ..., e_k^{\mathbf{H}}\} = E^{\mathbf{H}}$, for $i \in \{m+1, ..., m+k\}$, we can define $D_i^{\mathrm{Bl}(\mathbf{H})}$ to be the affine set given by the affine constraints

$$\begin{pmatrix} 0 & -I \\ 0 & \lambda^T_{\pi_1(e^{\mathbf{H}}_{i-m})} + \lambda^T_{\pi_2(e^{\mathbf{H}}_{i-m})} \\ 0 & -\lambda^T_{\pi_1(e^{\mathbf{H}}_{i-m})} - \lambda^T_{\pi_2(e^{\mathbf{H}}_{i-m})} \\ 0 & I \\ A^{\mathbf{H}}_{e^{\mathbf{H}}_{i-m}} & 0 \end{pmatrix} \begin{pmatrix} x \\ q \end{pmatrix} + \begin{pmatrix} \lambda_{\pi_1(e^{\mathbf{H}}_{i-m})} + \lambda_{\pi_2(e^{\mathbf{H}}_{i-m})} \\ -1 \\ 1 \\ 0 \\ a^{\mathbf{H}}_{e^{\mathbf{H}}_{i-m}} \end{pmatrix} \geq 0.$$

$\underline{E^{\mathrm{Bl}(\mathbf{H})}}$: Again letting $\{e_1^{\mathbf{H}}, ..., e_k^{\mathbf{H}}\} = E^{\mathbf{H}}$, define

$$E^{\mathrm{Bl}(\mathbf{H})} = \{e_1^{\mathrm{Bl}(\mathbf{H})}, ..., e_{2k}^{\mathrm{Bl}(\mathbf{H})}\},$$

where

$$\begin{aligned} e_i^{\mathrm{Bl}(\mathbf{H})} &= ((\pi_1(e_i^{\mathbf{H}}), i+m), (\pi_3(e_i^{\mathbf{H}}), \pi_1(e_i^{\mathbf{H}})), \\ e_{i+k}^{\mathrm{Bl}(\mathbf{H})} &= ((i+m, \pi_2(e_i^{\mathbf{H}})), (\pi_2(e_i^{\mathbf{H}}), \pi_4(e_i^{\mathbf{H}})), \end{aligned}$$

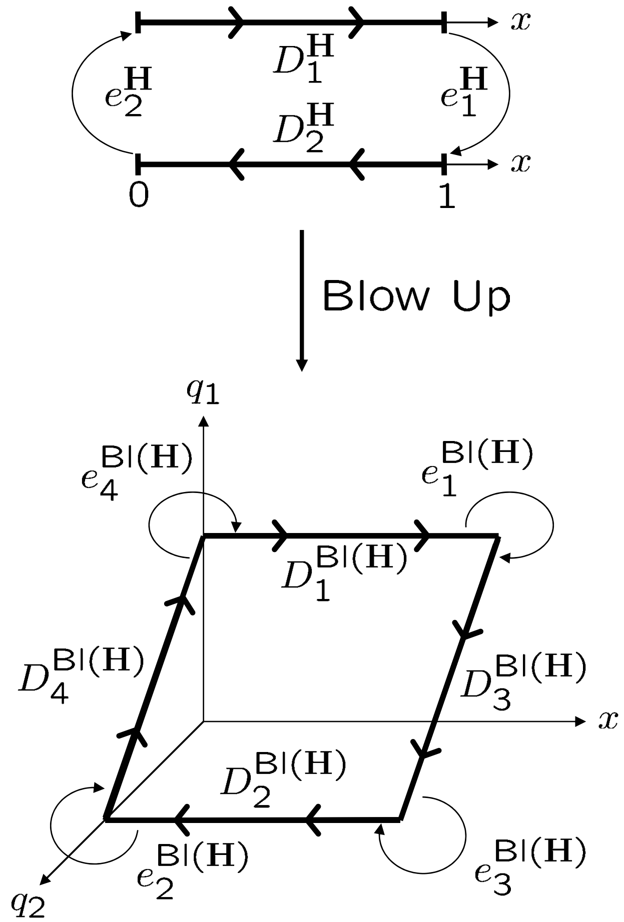

Fig. 1. An illustration of the blow up construction; in this case, the blow up construction applied to the thermostat, cf. Example 4.1.

for $1 \leq i \leq k$. By construction it follows that

$$T_{e_1^{\mathrm{Bl}(\mathbf{H})}} = \cdots = T_{e_{2k}^{\mathrm{Bl}(\mathbf{H})}} = \mathrm{id}.$$

$\underline{V^{\mathrm{Bl}(\mathbf{H})}}$: Write $\tilde{x} = (x, q) \in \mathbb{R}^{n+m}$, and define

$$V_i^{\mathrm{Bl}(\mathbf{H})}(\tilde{x}) = \begin{pmatrix} V_i^{\mathbf{H}}(x) \\ 0 \end{pmatrix}$$

for $1 \leq i \leq m$, and

$$V_i^{\mathrm{Bl}(\mathbf{H})}(\tilde{x}) = \begin{pmatrix} 0 \\ \lambda_{\pi_2(e^{\mathbf{H}}_{i-m})} - \lambda_{\pi_1(e^{\mathbf{H}}_{i-m})} \end{pmatrix}$$

for $m+1 \leq i \leq m+k$. To avoid confusion, let $\psi_i(t, \tilde{x})$ be the solution to $V_i^{\mathrm{Bl}(\mathbf{H})}$ for $\tilde{x} \in D_i^{\mathrm{Bl}(\mathbf{H})}$.

Example 4.1 (Thermostat): Consider the hybrid system referred to as the *thermostat*, $\mathbf{T} = \{Q^{\mathbf{T}}, D^{\mathbf{T}}, E^{\mathbf{T}}, V^{\mathbf{T}}\}$. Let $Q^{\mathbf{T}} = \{1, 2\}$, and $D_1^{\mathbf{T}} = D_2^{\mathbf{T}}$ be given by the affine constraints

$$A_i^{\mathbf{T}} x + a_i^{\mathbf{T}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \geq 0, \quad i = 1, 2.$$

We have $E^{\mathbf{T}} = \{e_1^{\mathbf{T}}, e_2^{\mathbf{T}}\}$, with edges $e_1^{\mathbf{T}} = ((1,2),(1,1))$ and $e_2^{\mathbf{T}} = ((2,1),(2,2))$; by (1), this implies that

$$\begin{aligned}\mathrm{Source}(e_1^{\mathbf{T}}) = \mathrm{Target}(e_1^{\mathbf{T}}) = \{x = 1\},\\ \mathrm{Source}(e_2^{\mathbf{T}}) = \mathrm{Target}(e_2^{\mathbf{T}}) = \{x = 0\}.\end{aligned}$$

For $V^{\mathbf{T}} = \{V_1^{\mathbf{T}}, V_2^{\mathbf{T}}\}$, $V_1^{\mathbf{T}}(x) = 1$ and $V_2^{\mathbf{T}}(x) = -1$.

Applying the blow up construction to $\mathbf{T}$, yields $\mathrm{Bl}(\mathbf{T})$ (as seen in Figure 1). We have $Q^{\mathrm{Bl}(\mathbf{T})} = \{1,2,3,4\}$, and

$$\begin{aligned}D_1^{\mathrm{Bl}(\mathbf{T})} &= \{x \in [0,1],\ q_1 = 1,\ q_2 = 0\},\\ D_2^{\mathrm{Bl}(\mathbf{T})} &= \{x \in [0,1],\ q_1 = 0,\ q_2 = 1\},\\ D_3^{\mathrm{Bl}(\mathbf{T})} &= \{x = 1,\ q_1 + q_2 = 1,\ q_1, q_2 \geq 0\},\\ D_4^{\mathrm{Bl}(\mathbf{T})} &= \{x = 0,\ q_1 + q_2 = 1,\ q_1, q_2 \geq 0\}.\end{aligned}$$

The construction also yields the edges

$$\begin{aligned}e_1^{\mathrm{Bl}(\mathbf{T})} = ((1,3),(1,1)), \qquad e_3^{\mathrm{Bl}(\mathbf{T})} = ((3,2),(2,1)),\\ e_2^{\mathrm{Bl}(\mathbf{T})} = ((2,4),(2,2)), \qquad e_4^{\mathrm{Bl}(\mathbf{T})} = ((4,1),(1,2)).\end{aligned}$$

Applying (1) to the affine constraints determining $D_i^{\mathrm{Bl}(\mathbf{T})}$ as given by the blow up construction (which we do not state due to space constraints) implies that the transition maps are the identity. Finally, there are the vector fields

$$V_1^{\mathrm{Bl}(\mathbf{T})}(\tilde{x}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad V_3^{\mathrm{Bl}(\mathbf{T})}(\tilde{x}) = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

$$V_2^{\mathrm{Bl}(\mathbf{T})}(\tilde{x}) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad V_4^{\mathrm{Bl}(\mathbf{T})}(\tilde{x}) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Note that this example illustrates some of the additional desirable properties that $\mathrm{Bl}(\mathbf{H})$ can possess. In particular, while $\mathbf{T}$ is not a switched system (because the interiors of the domains overlap and so the history of the system is needed for analysis), the blow up of $\mathbf{T}$ is a switched system (since the interiors of the domains do not overlap and so the history of the system is not needed for analysis).

V. Relationships between $\mathbf{H}$ and $\mathrm{Bl}(\mathbf{H})$

In this section, several relationships between $\mathbf{H}$ and $\mathrm{Bl}(\mathbf{H})$ are established. These are important in that they show that in some sense the qualitative behavior of $\mathbf{H}$ and $\mathrm{Bl}(\mathbf{H})$ are the same. More specifically, it is shown that there is an injective map from $S(\mathbf{H})$ to $S(\mathrm{Bl}(\mathbf{H}))$ and it is given explicitly. This is used to establish a bijection between $\mathscr{O}^{\mathbf{H}}$ and $\mathscr{O}^{\mathrm{Bl}(\mathbf{H})}$ which is again given explicitly. The constructive nature of the proofs of the propositions in the section is essential for the proof of $\mathscr{P}$-stability equivalence carried out in the following section.

5.1: To determine a map between $D_i^{\mathbf{H}}$ and $D_i^{\mathrm{Bl}(\mathbf{H})}$ consider the maps

$$D_i^{\mathbf{H}} \xrightarrow{\iota_i} \coprod_{i \in Q^{\mathbf{H}}} D_i^{\mathbf{H}} \xrightarrow{\Gamma} \bigcup_{i \in Q^{\mathbf{H}}} D_i^{\mathrm{Bl}(\mathbf{H})}$$

where

$$\begin{aligned}\iota_i(x) &= (x, i)\\ \Gamma((x,i)) &= \begin{pmatrix} x \\ \lambda_i \end{pmatrix} \in \mathbb{R}^{n+m}.\end{aligned}$$

Define $\Gamma_i = \Gamma \circ \iota_i$; this is the desired map from $D_i^{\mathbf{H}}$ to $D_i^{\mathrm{Bl}(\mathbf{H})}$. Note that $D_i^{\mathrm{Bl}(\mathbf{H})} = \Gamma_i(D_i^{\mathbf{H}})$ and Γ_i has a left inverse given by $\pi_x(x, \lambda_i) = x$.

Proposition 5.1: *There exists an injective map*

$$\Xi : S(\mathbf{H}) \longrightarrow S(\mathrm{Bl}(\mathbf{H}))$$

with a closed form solution.

Proof: Let $\chi = (\tau, \eta, \xi) \in S(\mathbf{H})$ and denote the image of Ξ by

$$\Xi(\chi) = (\Xi_1(\tau), \Xi_2(\eta), \Xi_3(\xi)).$$

Let d^{χ} and ρ^{χ} be the discrete state evolution and the evolution of edges for the execution χ: see Paragraph 3.2. Now let

$$\Lambda^{\Xi(\chi)} = \begin{cases} \{0, 1, ..., 2N^{\chi}\} & if \quad \chi \in S_0(\mathbf{H}) \\ \Lambda^{\chi} & \text{otherwise} \end{cases}$$

and define the map Ξ as:

$$\begin{aligned}\Xi_1(\tau)_i &= \begin{cases} \tau_{\frac{i}{2}} + \frac{i}{2} & if \quad \text{i even} \\ \tau_{\frac{i+1}{2}} + \frac{i-1}{2} & if \quad \text{i odd} \end{cases}\\ \Xi_2(\eta)_i &= \begin{cases} e^{\mathrm{Bl}(\mathbf{H})}_{\rho^{\chi}(\frac{i}{2})+k} & if \quad \text{i even} \\ e^{\mathrm{Bl}(\mathbf{H})}_{\rho^{\chi}(\frac{i+1}{2})} & if \quad \text{i odd} \end{cases}\\ \Xi_3(\xi)_i &= \begin{cases} \Gamma_{d^{\chi}(\frac{i}{2})}(\xi_{\frac{i}{2}}) & if \quad \text{i even} \\ \Gamma_{d^{\chi}(\frac{i-1}{2})}(\xi_{\frac{i+1}{2}}) & if \quad \text{i odd} \end{cases}\end{aligned}$$

where

$$\begin{aligned}e^{\mathrm{Bl}(\mathbf{H})}_{\rho^{\chi}(\frac{i}{2})+k} &= ((\rho^{\chi}(\frac{i}{2}) + m, \pi_2(\eta_{\frac{i}{2}})), (\pi_2(\eta_{\frac{i}{2}}), \pi_4(\eta_{\frac{i}{2}})))\\ e^{\mathrm{Bl}(\mathbf{H})}_{\rho^{\chi}(\frac{i+1}{2})} &= ((\pi_1(\eta_{\frac{i+1}{2}}), \rho^{\chi}(\frac{i+1}{2}) + m), (\pi_3(\eta_{\frac{i+1}{2}}), \pi_1(\eta_{\frac{i+1}{2}}))).\end{aligned}$$

The associated discrete state evolution and the evolution of edges for Ξ are given by

$$\begin{aligned}d^{\Xi(\chi)}(i) &= \begin{cases} d^{\chi}(\frac{i}{2}) & if \quad \text{i even} \\ \rho^{\chi}(\frac{i-1}{2}) + m & if \quad \text{i odd} \end{cases}\\ \rho^{\Xi(\chi)}(i) &= \begin{cases} \rho^{\chi}(\frac{i}{2}) + k & if \quad \text{i even} \\ \rho^{\chi}(\frac{i+1}{2}) & if \quad \text{i odd} \end{cases}\end{aligned}$$

To verify injectivity, a left inverse is need. Let $\tilde{\chi} = (\tilde{\tau}, \tilde{\eta}, \tilde{\xi})$ be a hybrid execution of $\mathrm{Bl}(\mathbf{H})$. Let

$$\Lambda^{\Omega(\tilde{\chi})} = \begin{cases} \{0, 1, ..., \frac{N^{\tilde{\chi}}}{2}\} & if \quad \tilde{\chi} \in S_0(\mathrm{Bl}(\mathbf{H}))\ \&\ N^{\tilde{\chi}} \text{ is even} \\ \{0, 1, ..., \frac{N^{\tilde{\chi}}-1}{2}\} & if \quad \tilde{\chi} \in S_0(\mathrm{Bl}(\mathbf{H}))\ \&\ N^{\tilde{\chi}} \text{ is odd} \\ \Lambda^{\tilde{\chi}} & \text{otherwise} \end{cases}$$

and define Ω by $\Omega(\tilde\chi)=(\Omega_1(\tilde\tau),\Omega_2(\tilde\eta),\Omega_3(\tilde\xi))$, where

$$\begin{aligned}\Omega_1(\tilde\tau)_i &= \tilde\tau_{2i}-i\\ \Omega_2(\tilde\eta)_i &= ((\pi_1(\tilde\eta_{2i-1}),\pi_2(\tilde\eta_{2i})),(\pi_3(\tilde\eta_{2i-1}),\pi_4(\tilde\eta_{2i-1})))\\ \Omega_3(\tilde\xi)_i &= \pi_x(\tilde\xi_{2i}).\end{aligned}$$

It can be verified easily that $\tau_i=\Omega_1(\Xi_1(\tau))_i$, $\eta_i=\Omega_2(\Xi_2(\eta))_i$ and $\xi_i=\Omega_3(\Xi_3(\xi))_i$, therefore $\Omega\circ\Xi=\mathrm{id}$, and Ω is the desired left inverse to Ξ. ■

Corollary 5.1: *There is a bijection of sets*

$$S_\infty(\mathbf{H}) \quad\longleftrightarrow\quad \begin{array}{c}\overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}\\ \|\\ \left\{\begin{array}{c}\tilde\chi\in S_\infty(\mathrm{Bl}(\mathbf{H}))\text{ with}\\ \tilde\xi_0\in D_i^{\mathrm{Bl}(\mathbf{H})}\text{ for } i\in Q^{\mathbf{H}}\end{array}\right\}\end{array}$$

Proof: It only needs to be verified that Ω is right the inverse to Ξ when restricted to $\overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}$. This can be verified by using the properties inherent to $\mathrm{Bl}(\mathbf{H})$ due to its construction; space constraints require omission of the details. ■

5.2: The case given in Corollary 5.1 will be of the most interest. For $\tilde\chi\in\overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}$ and for χ such that $\tilde\chi=\Xi(\chi)$ we can write $\psi^{\tilde\chi}$ as

$$\psi^{\tilde\chi}(t,\tilde\xi_0)=\frac{1}{\sum_{i\in\Lambda^{\tilde\chi}}a_i^{\tilde\chi}(t)}\left(\sum_{i\in\Lambda^{\tilde\chi}}a_{2i}^{\tilde\chi}(t)\psi_{d^{\tilde\chi}(2i)}(t-\tilde\tau_{2i},\tilde\xi_{2i})\right.$$
$$\left.+\sum_{i\in\Lambda_+^{\tilde\chi}}a_{2i-1}^{\tilde\chi}(t)\psi_{d^{\tilde\chi}(2i-1)}(t-\tilde\tau_{2i-1},\tilde\xi_{2i-1})\right)$$

where

$$\begin{aligned}\psi_{d^{\tilde\chi}(2i)}(t-\tilde\tau_{2i},\tilde\xi_{2i})&=\Gamma_{d^\chi(i)}(\varphi_{d^\chi(i)}(t-\tilde\tau_{2i},\xi_i))\\ \psi_{d^{\tilde\chi}(2i-1)}(t-\tilde\tau_{2i-1},\tilde\xi_{2i-1})&=(1-t+\tilde\tau_{2i-1})\Gamma_{d^\chi(i-1)}(\xi_i)\\ &\quad+(t-\tilde\tau_{2i})\Gamma_{d^\chi(i)}(\xi_i)\end{aligned}$$

Proposition 5.2: *There are the following bijections:*

$$\{\textbf{CSEP}\text{'s of }\mathbf{H}\}\quad\longleftrightarrow\quad\{\,\textbf{CSEP}\text{'s of }\mathrm{Bl}(\mathbf{H})\,\}$$

$$\{\textbf{CSPO}\text{'s of }\mathbf{H}\}\quad\longleftrightarrow\quad\{\,\textbf{CSPO}\text{'s of }\mathrm{Bl}(\mathbf{H})\,\}$$

$$\{\textbf{DSPO}\text{'s of }\mathbf{H}\}\quad\longleftrightarrow\quad\left\{\begin{array}{c}\textbf{MSPO}\text{'s of }\mathrm{Bl}(\mathbf{H})\text{ with}\\ \tilde\gamma\subseteq\bigcup_{i\in Q^{\mathrm{Bl}(\mathbf{H})}\setminus Q^{\mathbf{H}}}D_i^{\mathrm{Bl}(\mathbf{H})}\end{array}\right\}$$

$$\{\textbf{MSPO}\text{'s of }\mathbf{H}\}\quad\longleftrightarrow\quad\left\{\begin{array}{c}\textbf{MSPO}\text{'s of }\mathrm{Bl}(\mathbf{H})\text{ with}\\ \tilde\gamma\nsubseteq\bigcup_{i\in Q^{\mathrm{Bl}(\mathbf{H})}\setminus Q^{\mathbf{H}}}D_i^{\mathrm{Bl}(\mathbf{H})}\end{array}\right\}$$

Proof: The first and second bijections are clear. First let us verify the fourth bijection.

Now if γ is a **MSPO** of $\mathbf{H}$, the claim is that

$$\Upsilon(\gamma)=\bigcup_{\substack{\chi\,\in\,S_\infty(\mathbf{H})\\ \text{with }\xi_0\,\in\,\gamma}}\left(\bigcup_{i=1}^{K^\chi}\mathrm{ccl}\{\Gamma_{d^\chi(i-1)}(\xi_i),\Gamma_{d^\chi(i)}(\xi_i)\}\right.$$
$$\left.\cup\bigcup_{i=0}^{K^\chi}\{\Gamma_{d^\chi(i)}(\varphi_{d^\chi(i)}(t-\tau_i,\xi_i)):t\in[\tau_i,\tau_{i+1}]\}\right)$$

is a bijection, where "ccl" is the convex closure. Setting $\Upsilon^{-1}=\pi_x$, clearly $\Upsilon^{-1}\circ\Upsilon=\mathrm{id}$.

To verify that $\Upsilon\circ\Upsilon^{-1}=\mathrm{id}$, let $\tilde\gamma$ be a **MSPO** of $\mathrm{Bl}(\mathbf{H})$ with $\tilde\gamma\nsubseteq\bigcup_{i\in Q^{\mathrm{Bl}(\mathbf{H})}\setminus Q^{\mathbf{H}}}D_i^{\mathrm{Bl}(\mathbf{H})}$. Consider an execution $\tilde\chi\in\overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}$ with $\psi^{\tilde\chi}(t,\tilde\xi_0)\subseteq\tilde\gamma$. By Corollary 5.1 there exists a χ with $\tilde\chi=\Xi(\chi)$ and $K^{\tilde\chi}=2K^\chi$. Referring to Paragraph 5.2 there are the following relations

$$\begin{aligned}&\{\psi^{\tilde\chi}(t,\tilde\xi_0):t\in\mathbb{R}^+\}\\
&=\bigcup_{i=0}^{K^{\tilde\chi}}\{\psi_{d^{\tilde\chi}(i)}(t-\tilde\tau_i,\tilde\xi_i):t\in[\tilde\tau_i,\tilde\tau_{i+1}]\}\\
&=\bigcup_{i=1}^{K^\chi}\{\psi_{d^{\tilde\chi}(2i-1)}(t-\tilde\tau_{2i-1},\tilde\xi_{2i-1}):t\in[\tilde\tau_{2i-1},\tilde\tau_{2i}]\}\\
&\quad\cup\bigcup_{i=0}^{K^\chi}\{\psi_{d^{\tilde\chi}(2i)}(t-\tilde\tau_{2i},\tilde\xi_{2i}):t\in[\tilde\tau_{2i},\tilde\tau_{2i+1}]\}\\
&=\bigcup_{i=1}^{K^\chi}\{(1-t+\tilde\tau_{2i-1})\Gamma_{d^\chi(i-1)}(\xi_i)\\
&\qquad\qquad+(t-\tilde\tau_{2i})\Gamma_{d^\chi(i)}(\xi_i):t\in[\tilde\tau_{2i-1},\tilde\tau_{2i}]\}\\
&\quad\cup\bigcup_{i=0}^{K^\chi}\{\Gamma_{d^\chi(i)}(\varphi_{d^\chi(i)}(t-\tilde\tau_{2i},\xi_i)):t\in[\tilde\tau_{2i},\tilde\tau_{2i+1}]\}\\
&=\bigcup_{i=1}^{K^\chi}\mathrm{ccl}\{\Gamma_{d^\chi(i-1)}(\xi_i),\Gamma_{d^\chi(i)}(\xi_i)\}\\
&\quad\cup\bigcup_{i=0}^{K^\chi}\{\Gamma_{d^\chi(i)}(\varphi_{d^\chi(i)}(t-\tau_i,\xi_i)):t\in[\tau_i,\tau_{i+1}]\}.\end{aligned}$$

Now $\pi_x(\tilde\gamma)$ is a **MSPO** of $\mathbf{H}$, and we have

$$\begin{aligned}\tilde\gamma&=\bigcup_{\substack{\tilde\chi\,\in\,S_\infty(\mathrm{Bl}(\mathbf{H}))\\ \text{with }\tilde\xi_0\,\in\,\tilde\gamma}}\{\psi^{\tilde\chi}(t,\tilde\xi_0):t\in\mathbb{R}^+\}\\
&=\bigcup_{\substack{\tilde\chi\,\in\,\overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}\\ \text{with }\tilde\xi_0\,\in\,\tilde\gamma}}\{\psi^{\tilde\chi}(t,\tilde\xi_0):t\in\mathbb{R}^+\}\\
&=\bigcup_{\substack{\chi\,\in\,S_\infty(\mathbf{H})\\ \xi_0\,\in\,\pi_x(\tilde\gamma)}}\left(\bigcup_{i=1}^{K^\chi}\mathrm{ccl}\{\Gamma_{d^\chi(i-1)}(\xi_i),\Gamma_{d^\chi(i)}(\xi_i)\}\right.\\
&\qquad\left.\cup\bigcup_{i=0}^{K^\chi}\{\Gamma_{d^\chi(i)}(\varphi_{d^\chi(i)}(t-\tau_i,\xi_i)):t\in[\tau_i,\tau_{i+1}]\}\right)\\
&=\Upsilon\circ\Upsilon^{-1}(\tilde\gamma).\end{aligned}$$

which proves the fourth bijection.

To prove the third bijection, let x^* be a **DSPO** . The claim is that

$$\Upsilon(x^*) = \bigcup_{\substack{\chi \in S_\infty(\mathbf{H}) \\ \text{with } \xi_0 = x^*}} \bigcup_{i=1}^{K^\chi} \mathrm{ccl}\{\Gamma_{d^\chi(i-1)}(x^*), \Gamma_{d^\chi(i)}(x^*)\}$$

is bijective. It will be seen that this is a special case of the fourth bijection.

Let $\tilde{\gamma}$ be a **MSPO** of $\mathrm{Bl}(\mathbf{H})$ with $\tilde{\gamma} \subseteq \bigcup_{i \in Q^{\mathrm{Bl}(\mathbf{H})} \setminus Q^{\mathbf{H}}} D_i^{\mathrm{Bl}(\mathbf{H})}$. Again consider an execution $\tilde{\chi} \in \overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}$ with $\psi^{\tilde{\chi}}(t, \tilde{\xi}_0) \subseteq \tilde{\gamma}$; in this case $\tilde{\xi}_0 \in \partial D_i^{\mathrm{Bl}(\mathbf{H})}$ and $\tilde{\xi}_0 = \Gamma_{d^{\tilde{\chi}}(0)}(x^*)$ for $x^* = \pi_x(\tilde{\gamma})$ (which is a single point because $\tilde{\gamma}$ is connected). By referring to the construction of $\mathrm{Bl}(\mathbf{H})$ and Paragraph 5.2

$$\psi_{d^{\tilde{\chi}}(2i)}(t - \tilde{\tau}_{2i}, \tilde{\xi}_{2i}) \notin \bigcup_{i \in Q^{\mathrm{Bl}(\mathbf{H})} \setminus Q^{\mathbf{H}}} D_i^{\mathrm{Bl}(\mathbf{H})},$$

if $t \neq \tilde{\tau}_{2i}, \tilde{\tau}_{2i+1}$. Therefore, $\tilde{\tau}_{2i} = \tilde{\tau}_{2i+1}$. By Corollary 5.1, $\tilde{\chi} = \Xi(\chi)$ and for this χ, $\tilde{\tau}_{2i} = \tau_i = \tau_{i+1} = \tilde{\tau}_{2i+1}$. This gives

$$\begin{aligned} &\{\Gamma_{d^\chi(i)}(\varphi_{d^\chi(i)}(t - \tau_i, \xi_i)) : t \in [\tau_i, \tau_{i+1}]\} \\ &\qquad = \{\Gamma_{d^\chi(i)}(\xi_i)\} \in \mathrm{ccl}\{\Gamma_{d^\chi(i-1)}(\xi_i), \Gamma_{d^\chi(i)}(\xi_i)\}. \end{aligned}$$

Now since $\pi_x(\tilde{\gamma}) = x^*$ (again because $\tilde{\gamma}$ is connected) and $\tilde{\xi}_i \in \tilde{\gamma}$, $\xi_i = \pi_x(\tilde{\xi}_{2i}) = x^*$. Therefore,

$$\begin{aligned} \tilde{\gamma} &= \bigcup_{\substack{\tilde{\chi} \in \overline{S_\infty(\mathrm{Bl}(\mathbf{H}))} \\ \text{with } \tilde{\xi}_0 \in \tilde{\gamma}}} \{\psi^{\tilde{\chi}}(t, \tilde{\xi}_0) : t \in \mathbb{R}^+\} \\ &= \bigcup_{\substack{\chi \in S_\infty(\mathbf{H}) \\ \xi_0 = x^*}} \left(\bigcup_{i=1}^{K^\chi} \mathrm{ccl}\{\Gamma_{d^\chi(i-1)}(\xi_i), \Gamma_{d^\chi(i)}(\xi_i)\} \right. \\ &\qquad \left. \cup \bigcup_{i=0}^{K^\chi} \{\Gamma_{d^\chi(i)}(\varphi_{d^\chi(i)}(t - \tau_i, \xi_i)) : t \in [\tau_i, \tau_{i+1}]\} \right) \\ &= \bigcup_{\substack{\chi \in S_\infty(\mathbf{H}) \\ \xi_0 = x^*}} \bigcup_{i=1}^{K^\chi} \mathrm{ccl}\{\Gamma_{d^\chi(i-1)}(x^*), \Gamma_{d^\chi(i)}(x^*)\} \\ &= \Upsilon \circ \Upsilon^{-1}(\tilde{\gamma}). \end{aligned}$$

This completes the proof. ∎

Proposition 5.3: *There is a bijection*

$$\Upsilon : \mathscr{O}^{\mathbf{H}} \longrightarrow \mathscr{O}^{\mathrm{Bl}(\mathbf{H})}.$$

Proof: This follows from Proposition 5.2 and from the fact that $\mathrm{Bl}(\mathbf{H})$ has no **DSPO**'s . ∎

VI. Important Properties of $\mathrm{Bl}(\mathbf{H})$

In this section we prove the main results of this paper. The proofs of these theorems rely heavily on the construction of $\mathrm{Bl}(\mathbf{H})$ and the propositions established in the previous section, e.g., the proof that $\mathrm{Bl}(\mathbf{H})$ has no Zeno executions essentially follows from the construction of $\mathrm{Bl}(\mathbf{H})$.

Theorem 1: $\mathrm{Bl}(\mathbf{H})$ *has no Zeno executions.*

Proof: Suppose that $\mathrm{Bl}(\mathbf{H})$ had a Zeno execution $\tilde{\chi} \in S_\infty(\mathrm{Bl}(\mathbf{H}))$; without loss of generality let $\tilde{\chi} \in \overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}$. Then $\tilde{\chi} = \Xi(\chi)$ for some execution χ of $\mathbf{H}$. We are assuming that $\tilde{\chi}$ is Zeno, so $\Xi(\tau)_i \leq B$ for some integer B and all $i \in \Lambda^{\Xi(\chi)}$. But

$$\Xi(\tau)_{2B+2} = \tau_{\frac{2B+2}{2}} + \frac{2B+2}{2} \geq B + 1 > B$$

which gives a contradiction. ∎

Theorem 2: $\mathbf{H}$ *and* $\mathrm{Bl}(\mathbf{H})$ *are* $\mathscr{P}$*-stability equivalent.*

Proof: First consider the case when μ is a **CSEP** or **CSPO**. By Proposition 5.2, $\Upsilon(\mu)$ is also a **CSEP** or a **CSPO**. If $\mu \subset D_i^{\mathbf{H}}$ for $i \in Q^{\mathbf{H}}$, then $\Upsilon(\mu) \subset D_i^{\mathrm{Bl}(\mathbf{H})}$ (note that this does not include the degenerate case where $\mu = D_i^{\mathbf{H}}$, but the proof of this case is clear). Now pick δ such that $B_\delta(\mu) \subset D_i^{\mathbf{H}}$ and $B_\delta(\Upsilon(\mu)) \subset D_i^{\mathrm{Bl}(\mathbf{H})}$. There is an obvious bijection

$$\left\{ \begin{array}{c} \chi \in S(\mathbf{H}) \text{ with} \\ \xi_0 \in B_\delta(\mu) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \chi \in S(\mathrm{Bl}(\mathbf{H})) \text{ with} \\ \tilde{\xi}_0 \in B_\delta(\Upsilon(\mu)) \end{array} \right\}$$

Combining this bijection with the formula for $\Upsilon(\mu)$ given in the proof of Proposition 5.2,

$$\|\psi^{\tilde{\chi}}(t, \tilde{\xi}_0) - \Upsilon(\mu)\| = \|\varphi^\chi(t, \xi_0) - \mu\|,$$

which implies $\mathscr{P}$-stability equivalence.

Now consider the case where μ is a **DSPO** or a **MSPO**; in this case $\Upsilon(\mu)$ is given in the proof of Proposition 5.2. Without loss of generality, only the executions in $\overline{S_\infty(\mathbf{H})}$ and $\overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}$ need be considered. For every $\tilde{\chi} \in \overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}$, $\tilde{\chi} = \Xi(\chi)$. By Paragraph 5.2, for $t \in [\tilde{\tau}_{2i}, \tilde{\tau}_{2i+1}]$,

$$\begin{aligned} &\|\psi^{\tilde{\chi}}(t, \tilde{\xi}_0) - \Upsilon(\mu)\| \\ &\qquad = \|\psi_{d^{\tilde{\chi}}(2i)}(t - \tilde{\tau}_{2i}, \tilde{\xi}_{2i}) - \Upsilon(\mu)\| \\ &\qquad = \|\Gamma_{d^\chi(i)}(\varphi_{d^\chi(i)}(t - \tilde{\tau}_{2i}, \xi_i)) - \Upsilon(\mu)\| \\ &\qquad = \|\varphi_{d^\chi(i)}(t - \tilde{\tau}_{2i}, \xi_i) - \mu\| \\ &\qquad = \|\varphi_{d^\chi(i)}(t - \tau_i - i, \xi_i) - \mu\| \end{aligned} \tag{2}$$

and for $t \in [\tilde{\tau}_{2i-1}, \tilde{\tau}_{2i}]$,

$$\begin{aligned} &\|\psi^{\tilde{\chi}}(t, \tilde{\xi}_0) - \Upsilon(\mu)\| \\ &\qquad = \|\psi_{d^{\tilde{\chi}}(2i-1)}(t - \tilde{\tau}_{2i-1}, \tilde{\xi}_{2i-1}) - \Upsilon(\mu)\| \\ &\qquad = \|(1 - t + \tilde{\tau}_{2i-1})\Gamma_{d^\chi(i-1)}(\xi_i) \\ &\qquad\qquad + (t - \tilde{\tau}_{2i})\Gamma_{d^\chi(i)}(\xi_i) - \Upsilon(\mu)\| \\ &\qquad = \|\xi_i - \mu\|. \end{aligned} \tag{3}$$

Conversely, for $\chi \in S_\infty(\mathbf{H})$, and $\tilde{\chi} \in \overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}$ such that $\chi = \Omega(\tilde{\chi})$, for $t \in [\tau_i, \tau_{i+1}]$,

$$\begin{aligned} &\|\varphi^\chi(t, \xi_i) - \mu\| \\ &\qquad = \|\varphi_{d^\chi(i)}(t - \tau_i, \xi_i) - \mu\| \\ &\qquad = \|\psi_{d^{\tilde{\chi}}(2i)}(t - \tau_i + i, \tilde{\xi}_{2i}) - \Upsilon(\mu)\|. \end{aligned} \tag{4}$$

To show $\mathscr{P}$-stability equivalence, it must be shown that

$$\mu \text{ is } \mathscr{P}\text{-stable} \quad \Leftrightarrow \quad \Upsilon(\mu) \text{ is } \mathscr{P}\text{-stable}$$

for $\mathscr{P}$ = **LYP**, **ASY**, or **EXP**. Throughout the rest of the proof, let $B_\delta(\mu)$ and $B_\delta(\Upsilon(\mu))$ be sufficiently small neighborhoods such that $\pi_x(B_\delta(\Upsilon(\mu))) = B_\delta(\mu)$, and consider only $\chi \in \overline{S_\infty(\mathbf{H})}$ with $\xi_0 \in B_\delta(\mu)$ and $\tilde{\chi} \in \overline{S_\infty(\mathrm{Bl}(\mathbf{H}))}$ with $\tilde{\xi}_0 \in B_\delta(\Upsilon(\mu))$.

$\mathscr{P}$ = **LYP**: ($\Leftrightarrow$) Follows from (2), (3) and (4).

$\mathscr{P}$ = **ASY**: It suffices to show that

$$\begin{array}{c} \lim_{i\to\infty} \|\varphi^{\chi}(\tau_i,\xi_0) - \mu\| \to 0 \\ \Updownarrow \\ \lim_{i\to\infty} \|\psi^{\tilde{\chi}}(\tilde{\tau}_i,\tilde{\xi}_0) - \Upsilon(\mu)\| \to 0 \end{array}$$

($\Rightarrow$) Because

$$\|\tilde{\xi}_{2i-1} - \Upsilon(\mu)\| = \|\tilde{\xi}_{2i} - \Upsilon(\mu)\| = \|\xi_i - \mu\|,$$

it follows that

$$\begin{aligned} &\lim_{i\to\infty} \|\psi^{\tilde{\chi}}(\tilde{\tau}_{2i},\xi_0) - \Upsilon(\mu)\| \\ &\qquad = \lim_{i\to\infty} \|\psi^{\tilde{\chi}}(\tilde{\tau}_{2i-1},\xi_0) - \Upsilon(\mu)\| \\ &\qquad = \lim_{i\to\infty} \|\xi_i - \mu\| \\ &\qquad = \lim_{i\to\infty} \|\varphi^{\chi}(\tau_i,\xi_0) - \mu\| \to 0, \end{aligned}$$

which implies the result.

$$(\Leftarrow) \qquad \begin{aligned} \lim_{i\to\infty} \|\varphi^{\chi}(\tau_i,\xi_0) - \mu\| &= \lim_{i\to\infty} \|\xi_i - \mu\| \\ &= \lim_{i\to\infty} \|\tilde{\xi}_{2i} - \Upsilon(\mu)\| \\ &= \lim_{i\to\infty} \|\psi^{\tilde{\chi}}(\tilde{\tau}_{2i},\tilde{\xi}_0) - \Upsilon(\mu)\| \to 0. \end{aligned}$$

$\mathscr{P}$ = **EXP**: ($\Leftarrow$) Suppose that

$$\|\psi^{\tilde{\chi}}(t,\tilde{\xi}_0) - \Upsilon(\mu)\| \leq M e^{-\alpha t}\|\tilde{\xi}_0 - \Upsilon(\mu)\|, \quad t \in \mathfrak{T}^{\tilde{\chi}}.$$

Then for all $t \in [\tau_i, \tau_{i+1}]$,

$$\begin{aligned} &\|\varphi_{d^{\chi}(i)}(t - \tau_i, \xi_i) - \mu\| \\ &\qquad = \|\psi_{d^{\tilde{\chi}}(2i)}(t - \tau_i + i, \tilde{\xi}_{2i}) - \Upsilon(\mu)\| \\ &\qquad \leq M e^{-\alpha(t+i)}\|\tilde{\xi}_0 - \Upsilon(\mu)\| \\ &\qquad \leq M e^{-\alpha t}\|\xi_0 - \mu\|. \end{aligned}$$

($\Rightarrow$) Suppose that

$$\|\varphi^{\chi}(t,\xi_0) - \mu\| \leq M e^{-\alpha t}\|\xi_0 - \mu\|, \quad t \in \mathfrak{T}^{\chi},$$

and define

$$\begin{aligned} \beta &= \min_{i\in\Lambda^{\chi}} \beta_i, \\ \beta_i &= \frac{-\log \frac{\|\varphi^{\chi}(\tau_i,\xi_0)-\mu\|}{M\|\xi_0-\mu\|}}{i - \frac{1}{\alpha}\log \frac{\|\varphi^{\chi}(\tau_i,\xi_0)-\mu\|}{M\|\xi_0-\mu\|}}. \end{aligned}$$

It can be verified that $\beta > 0$, $\beta_i \leq \alpha$, and for all $t \geq \tilde{\tau}_{2i}$,

$$e^{-\alpha(t-i)} \;\leq\; e^{-\beta_i t} \;\leq\; e^{-\beta t}.$$

From this the result follows since for $t \in [\tilde{\tau}_{2i}, \tilde{\tau}_{2i+1}]$, by (2),

$$\begin{aligned} &\|\psi^{\tilde{\chi}}(t,\tilde{\xi}_0) - \Upsilon(\mu)\| \\ &\qquad = \|\varphi_{d^{\chi}(i)}(t - \tau_i - i, \xi_i) - \mu\| \\ &\qquad \leq M e^{-\alpha(t-i)}\|\xi_0 - \mu\| \\ &\qquad \leq M e^{-\beta_i t}\|\xi_0 - \mu\| \\ &\qquad \leq M e^{-\beta t}\|\tilde{\xi}_0 - \Upsilon(\mu)\| \end{aligned}$$

and for all $t \in [\tilde{\tau}_{2i-1}, \tilde{\tau}_{2i}]$,

$$\begin{aligned} &\|\psi^{\tilde{\chi}}(t,\tilde{\xi}_0) - \Upsilon(\mu)\| \\ &\qquad = \|\tilde{\xi}_{2i} - \Upsilon(\mu)\| \\ &\qquad \leq M e^{-\beta\tilde{\tau}_{2i}}\|\tilde{\xi}_0 - \Upsilon(\mu)\| \\ &\qquad \leq M e^{-\beta t}\|\tilde{\xi}_0 - \Upsilon(\mu)\|. \end{aligned}$$

Therefore,

$$\|\psi^{\tilde{\chi}}(t,\tilde{\xi}_0) - \Upsilon(\mu)\| \leq M e^{-\beta t}\|\tilde{\xi}_0 - \Upsilon(\mu)\|, \quad t \in \mathfrak{T}^{\tilde{\chi}}.$$

■

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