H-Categories and Graphs

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Abstract

H-categories are an essential ingredient of categorical models for hybrid systems. In this note, we consider oriented H-categories and form the category of these categories, Hcat. The main result is that there is an isomorphism of categories:

$$\mathsf{Hcat} \cong \mathsf{Grph}\,.$$

The proof of this fact is constructive in nature, i.e., it is shown how to obtain a graph from and H-category and an H-category from a graph.

1. Introduction

Small categories play a critical role in the study of diagrams in a category, i.e., one often considerers the the functor category T^{C} whose objects are all functors $\boldsymbol{F} : \mathsf{C} \to \mathsf{T}$, for a small category C. Similarly, diagrams can also be defined by a graph whose vertices index a collection of objects in T and edges index a collection of morphisms in T. These two notions of "diagrams" in a category are related via functors from Grph to Cat and Cat to Grph which form an adjoint pair [7]. The disadvantage to this construction is that the creation of a small category from a graph is achieved by adding information to the graph (in the form of paths in the graph); it may be the case that this added information is unwanted and/or unneeded. This motivates the creation of a small category from a graph in which the resulting category is the "same" as the original graph, i.e., we would like to find a subcategory of Cat that is isomorphic to Grph. The purpose of this note is to find this subcategory.

We start by defining a specific type of small category termed an *H-category* and denoted by H. This is a small category in which every diagram has the form:

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \quad \cdots \quad \bullet \longleftarrow \bullet \longrightarrow \bullet$$

That is, an H-category has as its basic atomic unit a diagram of the form:

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

and any other diagram in this category must be obtainable by gluing such atomic units along the target of a morphism (and not the source). We can orient an H-category by, once and for all, picking a labeling of its morphisms. This allows us to form the category of all H-categories Hcat, which is a subcategory of Cat.

Given an oriented graph, we are able to explicitly construct and oriented H-category, and vice-versa, i.e. we can define functors from Hcat to Grph and from Grph to Hcat. The main result of this paper is a proof of the existence of the following isomorphism of categories:

$$\mathsf{Grph} \cong \mathsf{Hcat}.$$

This result has implications in the area of hybrid systems—systems that display both discrete and continuous behavior. Recently, a categorical model for these system has been proposed (cf. [1]-[5]); the starting point for this theory is the notion of a hybrid object over a category T. This is a diagram in T, $\boldsymbol{F}: \mathsf{H} \to \mathsf{T}$, in which the indexing small category is an H-category—no other small category would be suitable for this theory, and one could not work with graphs. The main result of this note is the completely characterization of the indexing objects of hybrid objects.

2. H-categories

In this section, we introduction the notion of a oriented H-category.

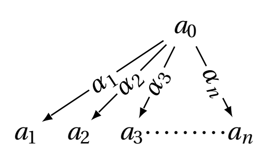
2.1. H-categories. An *H-category* is a small category H satisfying the following conditions:

1. Every object in H is either the source of a non-identity morphism in H or the target of a non-identity morphism but never both, i.e., for every diagram:

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n$$

   in H, all but one morphism must be the identity (the longest chain of composable non-identity morphisms is of length one).

2. If an object in H is the source of a non-identity morphism, then it is the source of exactly two non-identity morphisms, i.e., for every diagram in H of the form:



   either all of the morphisms are the identity or two and only two morphisms are not the identity.

Note that the category of H-categories forms a full subcategory of the category of small categories.

2.2. Important objects in H-categories. Let H be a H-category. We use $\mathsf{Ob}(\mathsf{H})$ to denote the objects of H, $\mathsf{Mor}(\mathsf{H})$ to denote its morphisms of H, and $\mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H})$ to denote the set of non-identity morphisms of H. For a morphism $\alpha : a \to b$ in H, its domain (or source) is denoted by $\mathsf{dom}(\alpha) = a$ and its codomain (or target) is denoted by $\mathsf{cod}(\alpha) = b$. For small H-categories, there are two sets of objects that are of particular interest; these are subsets of the set $\mathsf{Ob}(\mathsf{H})$. The first of these is called the *edge set of* H, is denoted by $\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$, and is defined to be:

$$\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) = \{a \in \mathsf{Ob}(\mathsf{H}) : \ a = \mathsf{dom}(\alpha),\ a = \mathsf{dom}(\beta),\ \alpha, \beta \in \mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H}),\ \alpha \neq \beta\}.$$

That is, for all $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$ there are two and only two morphisms (which are not the identity) $\alpha, \beta \in \mathsf{Mor}(\mathsf{H})$ such that $a = \mathsf{dom}(\alpha)$ and $a = \mathsf{dom}(\beta)$, so we denote these morphisms by α_a and β_a (we will pick, once and for all, a labeling of the morphisms in H in this way—this will define an orientation for H). Conversely, given a morphism $\gamma \in \mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H})$, there exists a unique $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$ such that $\gamma = \alpha_a$ or $\gamma = \beta_a$. The symbol $\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}()$ is used because every object $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$ sits in a diagram of the form:

$$b = \mathsf{cod}(\alpha_a) \xleftarrow{\ \alpha_a\ } \mathsf{dom}(\alpha_a) = a = \mathsf{dom}(\beta_a) \xrightarrow{\ \beta_a\ } \mathsf{cod}(\beta_a) = c$$

called a *bac-diagram*. Note that giving all diagrams of this form (of which there is one for each $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$) gives all the objects in H, i.e., every object of H is the target of α_a or β_a, or their source, for some $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$. More specifically, we can define the *vertex set of* H by

$$\mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}) = \left(\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})\right)^c$$

where here $\left(\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})\right)^c$ is the complement of $\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$ in the set $\mathsf{Ob}(\mathsf{H})$. It follows by definition that

$$\begin{aligned} \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) \cap \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}) &= \emptyset \\ \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) \cup \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}) &= \mathsf{Ob}(\mathsf{H}). \end{aligned}$$

2.3. Orienting H-categories. We can orient an H-category by picking a specific labeling of its morphisms. Specifically, we define an *orientation* of an H-category H as a pair of maps (α, β) between sets:

$$\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathsf{Mor}_{\not{\mathsf{id}}}(\mathsf{H})$$

such that for every $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$, there is a bac-diagram in H:

$$b \xleftarrow{\ \alpha_a\ } a \xrightarrow{\ \beta_a\ } c$$

We say that an H-category is *oriented*, or that it is an *oriented H-category*, if it has been given an orientation.

An orientation of H defines a partition of the non-identity morphisms of H:

$$\begin{aligned}\mathsf{Mor}_{\alpha}(\mathsf{H}) &:= \{\gamma \in \mathsf{Mor}_{\not{\mathrm{id}}}(\mathsf{H}) : \gamma = \alpha_a \text{ for some } a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})\} \\ \mathsf{Mor}_{\beta}(\mathsf{H}) &:= \{\gamma \in \mathsf{Mor}_{\not{\mathrm{id}}}(\mathsf{H}) : \gamma = \beta_a \text{ for some } a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})\}\end{aligned}$$

wherein it follows that:

$$\begin{aligned}\mathsf{Mor}_{\alpha}(\mathsf{H}) \cap \mathsf{Mor}_{\beta}(\mathsf{H}) &= \emptyset \\ \mathsf{Mor}_{\alpha}(\mathsf{H}) \cup \mathsf{Mor}_{\beta}(\mathsf{H}) &= \mathsf{Mor}_{\not{\mathrm{id}}}(\mathsf{H}).\end{aligned}$$

We will always assume that a given H-category has an orientation.

Define the category of oriented H-categories, Hcat, to have objects H-categories. A morphism between two oriented H-categories, H and H' (with orientations (α, β) and (α', β'), respectively), is a functor $\boldsymbol{F} : \mathsf{H} \to \mathsf{H}'$ such that the following diagrams:

$$\begin{array}{ccc} \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) & \xrightarrow{\boldsymbol{F}} & \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}') \\ \downarrow \alpha & & \downarrow \alpha' \\ \mathsf{Mor}_{\not{\mathrm{id}}}(\mathsf{H}) & \xrightarrow{\boldsymbol{F}} & \mathsf{Mor}_{\not{\mathrm{id}}}(\mathsf{H}') \end{array} \qquad \begin{array}{ccc} \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) & \xrightarrow{\boldsymbol{F}} & \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}') \\ \downarrow \beta & & \downarrow \beta' \\ \mathsf{Mor}_{\not{\mathrm{id}}}(\mathsf{H}) & \xrightarrow{\boldsymbol{F}} & \mathsf{Mor}_{\not{\mathrm{id}}}(\mathsf{H}') \end{array} \tag{1}$$

commute. By requiring these diagrams to commute, we mean that if $a \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$ with corresponding bac-diagram:

$$b \xleftarrow{\ \alpha_a\ } a \xrightarrow{\ \beta_a\ } c$$

then there is a corrisponding bac-diagram:

$$\boldsymbol{F}(b) \xleftarrow{\ \boldsymbol{F}(\alpha_a) = \alpha'_{\boldsymbol{F}(a)}\ } \boldsymbol{F}(a) \xrightarrow{\ \boldsymbol{F}(\beta_a) = \beta'_{\boldsymbol{F}(a)}\ } \boldsymbol{F}(c)$$

where $\boldsymbol{F}(a) \in \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}')$.

3. Oriented H-categories and Oriented Graphs

We not turn our attention to the association of an H-category to a graph, and vice-versa.

3.1. Oriented Graphs. An oriented graph is a pair $\Gamma = (Q, E)$, where Q is a set of vertices and E is a set of edges, together with a pair of functions s and t

$$E \overset{\mathsf{s}}{\underset{\mathsf{t}}{\rightrightarrows}} Q$$

called the source and target functions: for $e \in E$, $\mathsf{s}(e)$ is the source of e and $\mathsf{t}(e)$ is the target of e.

A morphism of graphs is a pair $D = (D_Q, D_E) : \Gamma = (Q, E) \to \Gamma' = (Q', E')$, where $D_Q : Q \to Q'$ and $D_E : E \to E'$, such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{D_E} & E' \\ {\scriptstyle \mathsf{s}}\downarrow & & \downarrow{\scriptstyle \mathsf{s}'} \\ Q & \xrightarrow{D_Q} & Q' \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{D_E} & E' \\ {\scriptstyle \mathsf{t}}\downarrow & & \downarrow{\scriptstyle \mathsf{t}'} \\ Q & \xrightarrow{D_Q} & Q' \end{array} \tag{2}$$

The category of graphs, Grph, has as objects oriented graphs and as morphisms morphisms of graphs.

3.2. Oriented H-categories from oriented graphs. Given an oriented graph, $\Gamma = (Q, E)$, we can associate to this graph an (oriented) H-category H_Γ. First, we define the objects of H_Γ by using:

$$\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}_\Gamma) := E, \qquad \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}_\Gamma) := Q.$$

and so $\mathsf{Ob}(\mathsf{H}_\Gamma) = \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}_\Gamma) \cup \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}_\Gamma)$. To introduce the morphisms of H_Γ we define, for every $e \in E$, morphisms:

$$\mathsf{s}(e) \xleftarrow{\alpha_e} e \xrightarrow{\beta_e} \mathsf{t}(e)$$

and we define:

$$\mathsf{Mor}_\alpha(\mathsf{H}_\Gamma) := \{\alpha_e : e \to \mathsf{s}(e)\}_{e \in E}, \qquad \mathsf{Mor}_\beta(\mathsf{H}_\Gamma) := \{\beta_e : e \to \mathsf{t}(e)\}_{e \in E}$$

so $\mathsf{Mor}_{\not\mathrm{id}}(\mathsf{H}_\Gamma) = \mathsf{Mor}_\alpha(\mathsf{H}_\Gamma) \cup \mathsf{Mor}_\beta(\mathsf{H}_\Gamma)$. We complete the description of H_Γ by defining an identity morphism on each object of H_Γ. Note that in the definition of H_Γ, we gave it a canonical orientation; namely, (α, β) were α_e and β_e are defined as above for every $e \in E$.

Given a morphism $D : \Gamma \to \Gamma'$, we can define a functor $\boldsymbol{F}_D : \mathsf{H}_\Gamma \to \mathsf{H}_{\Gamma'}$ operating on objects as follows:

$$\boldsymbol{F}_D(a) := \begin{cases} D_E(a) & \text{if} \quad a \in E = \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}_\Gamma) \\ D_Q(a) & \text{if} \quad a \in Q = \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}_\Gamma) \end{cases} \tag{3}$$

and operating on morphisms as follows:

$$\boldsymbol{F}_D(\gamma) := \begin{cases} \alpha'_{\boldsymbol{F}_D(e)} : \boldsymbol{F}_D(e) \to \mathsf{s}(\boldsymbol{F}_D(e)) & \text{if} \quad \gamma = \alpha_e \in \mathsf{Mor}_\alpha(\mathsf{H}_\Gamma) \\ \beta'_{\boldsymbol{F}_D(e)} : \boldsymbol{F}_D(e) \to \mathsf{t}(\boldsymbol{F}_D(e)) & \text{if} \quad \gamma = \beta_e \in \mathsf{Mor}_\beta(\mathsf{H}_\Gamma) \end{cases} \tag{4}$$

Of course, $\boldsymbol{F}_D$ is defined on identity morphisms in the obvious fashion: $\boldsymbol{F}_D(\mathrm{id}_a) := \mathrm{id}_{\boldsymbol{F}_D(a)}$. Note that $\boldsymbol{F}_D$ is a valid morphism of H-categories; (2) commutes because (1) commutes.

The method of associating an oriented H-category to a graph defines a functor:

$$\begin{array}{rcl} \boldsymbol{\Gamma} : \mathsf{Grph} & \longrightarrow & \mathsf{Hcat} \\ \Gamma & \mapsto & \boldsymbol{\Gamma}(\Gamma) := \mathsf{H}_\Gamma\,. \end{array} \tag{5}$$

We can now introduce the inverse of this construction.

3.3. Oriented graphs from oriented H-categories. Given an oriented H-category H (that is, we have functions α and β), we can obtain an oriented graph from this H-category,

$$\Gamma_{\mathsf{H}} = (Q_{\mathsf{H}}, E_{\mathsf{H}}),$$

by defining:

$$Q_{\mathsf{H}} := \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}), \qquad E_{\mathsf{H}} := \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}),$$

with source and target functions:

$$E_{\mathsf{H}} = \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H}) \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H}) = Q_{\mathsf{H}}$$

defined by, for all $a \in E_{\mathsf{H}} = \mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})$, $\mathsf{s}(a) := \mathsf{cod}(\alpha_a)$ and $\mathsf{t}(a) := \mathsf{cod}(\beta_a)$. This in turn defines a functor:

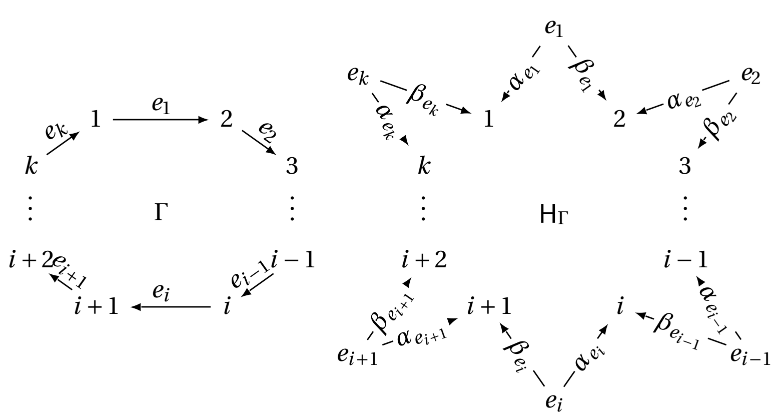
$$\begin{aligned} \boldsymbol{H} : \mathsf{Hcat} &\longrightarrow \mathsf{Grph} \\ \mathsf{H} &\longmapsto \boldsymbol{H}(\mathsf{H}) := \Gamma_{\mathsf{H}}. \end{aligned} \tag{6}$$

To finish the definition of this functor, we must define it on functors between H-categories. For a morphism between H-categories, $F : \mathsf{H} \to \mathsf{H}'$, define $\boldsymbol{H}(\boldsymbol{F}) := D_{\boldsymbol{F}}$ where $D_{\boldsymbol{F}}$ is the object function of $\boldsymbol{F}$ (that is, the functor $\boldsymbol{F}$ viewed as a function between the objects of H and the objects of H'). More specifically, we define:

$$D_{\boldsymbol{F}} = ((D_{\boldsymbol{F}})_Q, (D_{\boldsymbol{F}})_E) := (\boldsymbol{F}|_{Q_{\mathsf{H}}}, \boldsymbol{F}|_{E_{\mathsf{H}}}) = (\boldsymbol{F}|_{\mathsf{Ob}_{(\rightarrow\cdot\leftarrow)}(\mathsf{H})}, \boldsymbol{F}|_{\mathsf{Ob}_{(\leftarrow\cdot\rightarrow)}(\mathsf{H})})$$

Note that $D_{\boldsymbol{F}}$ is a valid morphism of graphs; (1) commutes because (2) commutes.

3.4. Example.
The following diagram shows an oriented cycle graph, $\Gamma = C_k$, and the associated H-category H_Γ:



4. Main Result

In this section we give the main result of this note.

4.1. Theorem.
There is an isomorphism of categories:

$$\mathsf{Hcat} \cong \mathsf{Grph}$$

where this isomorphism is given by the functor $\boldsymbol{H} : \mathsf{Hcat} \to \mathsf{Grph}$ *with inverse* $\boldsymbol{\Gamma} : \mathsf{Grph} \to \mathsf{Hcat}$.

Proof. We first verify that $\boldsymbol{\Gamma} \circ \boldsymbol{H} = \mathbf{Id}_{\mathsf{Hcat}}$. On objects, this holds since, using the notation of the previous paragraphs:

$$\begin{aligned}
\mathsf{Mor}_{\cancel{\mathrm{id}}}\,(\boldsymbol{\Gamma} \circ \boldsymbol{H}(\mathsf{H})) &= \mathsf{Mor}_{\cancel{\mathrm{id}}}\,(\mathsf{H}_{\Gamma_{\mathsf{H}}}) \\
&= \{e \to \mathsf{s}(e)\}_{e \in E_{\mathsf{H}}} \cup \{e \to \mathsf{t}(e)\}_{e \in E_{\mathsf{H}}} \\
&= \{a \to \mathsf{cod}(\alpha_a)\}_{a \in \mathsf{Ob}_{(\leftarrow\cdot\to)}(H)} \cup \{a \to \mathsf{cod}(\beta_a)\}_{a \in \mathsf{Ob}_{(\leftarrow\cdot\to)}(H)} \\
&= \mathsf{Mor}_\alpha(\mathsf{H}) \cup \mathsf{Mor}_\beta(\mathsf{H}) \\
&= \mathsf{Mor}_{\cancel{\mathrm{id}}}\,(\mathsf{H}).
\end{aligned}$$

And, using the notation of the previous paragraphs:

$$\begin{aligned}
\mathsf{Ob}(\boldsymbol{\Gamma} \circ \boldsymbol{H}(\mathsf{H})) &= \mathsf{Ob}(\mathsf{H}_{\Gamma_{\mathsf{H}}}) \\
&= E_{\mathsf{H}} \cup Q_{\mathsf{H}} \\
&= \mathsf{Ob}_{(\to\cdot\leftarrow)}(\mathsf{H}) \cup \mathsf{Ob}_{(\leftarrow\cdot\to)}(\mathsf{H}) \\
&= \mathsf{Ob}(\mathsf{H})
\end{aligned}$$

Next, the identity morphisms of H and $\mathbf{H} \circ \boldsymbol{\Gamma}(\mathsf{H})$ are the same by definition.
Finally, for $\boldsymbol{F} : \mathsf{H} \to \mathsf{H}'$, on objects:

$$\begin{aligned}
\boldsymbol{\Gamma} \circ \boldsymbol{H}(\boldsymbol{F})(a) &= \boldsymbol{F}_{D_{\boldsymbol{F}}}(a) \\
&= \begin{cases} \boldsymbol{F}(a) & \text{if} \quad a \in E_{\mathsf{H}} = \mathsf{Ob}_{(\leftarrow\cdot\to)}(\mathsf{H}_{\Gamma_{\mathsf{H}}}) = \mathsf{Ob}_{(\leftarrow\cdot\to)}(\mathsf{H}) \\ \boldsymbol{F}(a) & \text{if} \quad a \in Q_{\mathsf{H}} = \mathsf{Ob}_{(\to\cdot\leftarrow)}(\mathsf{H}_{\Gamma_{\mathsf{H}}}) = \mathsf{Ob}_{(\to\cdot\leftarrow)}(\mathsf{H}) \end{cases} \\
&= \boldsymbol{F}(a).
\end{aligned}$$

And, on morphisms:

$$\begin{aligned}
\boldsymbol{\Gamma} \circ \boldsymbol{H}(\boldsymbol{F})(\gamma) &= \boldsymbol{F}_{D_{\boldsymbol{F}}}(\gamma) \\
&= \begin{cases} \alpha'_{\boldsymbol{F}_{D_{\boldsymbol{F}}}(e)} : \boldsymbol{F}_{D_{\boldsymbol{F}}}(e) \to \mathsf{s}(\boldsymbol{F}_{D_{\boldsymbol{F}}}(e)) & \text{if} \quad \gamma = \alpha_e \in \mathsf{Mor}_\alpha(\mathsf{H}_{\Gamma_{\mathsf{H}}}) \\ \beta'_{\boldsymbol{F}_{D_{\boldsymbol{F}}}(e)} : \boldsymbol{F}_{D_{\boldsymbol{F}}}(e) \to \mathsf{t}(\boldsymbol{F}_{D_{\boldsymbol{F}}}(e)) & \text{if} \quad \gamma = \beta_e \in \mathsf{Mor}_\beta(\mathsf{H}_{\Gamma_{\mathsf{H}}}) \end{cases} \\
&= \begin{cases} \alpha'_{\boldsymbol{F}(a)} : \boldsymbol{F}(a) \to \mathsf{s}(\boldsymbol{F}(a)) & \text{if} \quad \gamma = \alpha_a \in \mathsf{Mor}_\alpha(\mathsf{H}) \\ \beta'_{\boldsymbol{F}(a)} : \boldsymbol{F}(a) \to \mathsf{t}(\boldsymbol{F}(a)) & \text{if} \quad \gamma = \beta_a \in \mathsf{Mor}_\beta(\mathsf{H}) \end{cases} \\
&= \begin{cases} \boldsymbol{F}(\gamma) : \boldsymbol{F}(a) \to \mathsf{cod}(\boldsymbol{F}(\gamma)) & \text{if} \quad \gamma = \alpha_a \in \mathsf{Mor}_\alpha(\mathsf{H}) \\ \boldsymbol{F}(\gamma) : \boldsymbol{F}(a) \to \mathsf{cod}(\boldsymbol{F}(\gamma)) & \text{if} \quad \gamma = \beta_a \in \mathsf{Mor}_\beta(\mathsf{H}) \end{cases} \\
&= \boldsymbol{F}(\gamma).
\end{aligned}$$

This demonstrates that: $\boldsymbol{\Gamma} \circ \boldsymbol{H} = \mathbf{Id}_{\mathsf{Hcat}}$.

Next we verify that $\boldsymbol{H} \circ \boldsymbol{\Gamma} = \mathbf{Id}_{\mathsf{Grph}}$. For a graph Γ, we have

$$\begin{aligned} \boldsymbol{H} \circ \boldsymbol{\Gamma}(\Gamma) &= \Gamma_{\mathsf{H}_\Gamma} \\ &= (Q_{\mathsf{H}_\Gamma}, E_{\mathsf{H}_\Gamma}) \\ &= (\mathsf{Ob}_{(\to\cdot\leftarrow)}(\mathsf{H}_\Gamma), \mathsf{Ob}_{(\leftarrow\cdot\to)}(\mathsf{H}_\Gamma)) \\ &= (Q, E) = \Gamma \end{aligned}$$

Consider a morphism of graphs $D = (D_Q, D_E) : \Gamma \to \Gamma'$. If $q \in Q$, then

$$\begin{aligned} (\boldsymbol{H} \circ \boldsymbol{\Gamma}(D))_{Q_{\mathsf{H}_\Gamma}}(q) &= (\boldsymbol{H} \circ \boldsymbol{\Gamma}(D))_Q(q) \\ &= (D_{\boldsymbol{F}_D})_Q(q) \\ &= \boldsymbol{F}_D(q) \\ &= D_Q(q) \end{aligned}$$

And if $e \in E$,

$$\begin{aligned} (\boldsymbol{H} \circ \boldsymbol{\Gamma}(D))_{E_{\mathsf{H}_\Gamma}}(e) &= (\boldsymbol{H} \circ \boldsymbol{\Gamma}(D))_E(e) \\ &= (D_{\boldsymbol{F}_D})_E(q) \\ &= \boldsymbol{F}_D(e) \\ &= D_E(e) \end{aligned}$$

Therefore, $(\boldsymbol{H} \circ \boldsymbol{\Gamma}(D)) = D$, which completes the proof. □

4.2. Corollary.
There is an adjunction from Hcat *to* Grph, $\langle \boldsymbol{H}, \boldsymbol{\Gamma}, \phi \rangle$, *where*

$$\mathsf{Hcat} \underset{\boldsymbol{\Gamma}}{\overset{\boldsymbol{H}}{\rightleftarrows}} \mathsf{Grph}$$

and ϕ *is a function which assigns to every pair of objects* $\mathsf{H} \in \mathsf{Ob}(\mathsf{H})$ *and* $\Gamma \in \mathsf{Ob}(\mathsf{Grph})$ *a bijection of sets*

$$\phi_{\mathsf{H},\Gamma} : \mathsf{Hcat}(\mathsf{H}_\Gamma, \Gamma) \cong \mathsf{Grph}(\Gamma, \Gamma_{\mathsf{H}})$$

which is natural in H *and* Γ.

Proof. Defining the function ϕ is equivalent to defining the unit and counit of the adjunction. In this case, we define the counit and the unit to be the identity natural transformations. □

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