Affine Hybrid Systems

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Abstract. Affine hybrid systems are hybrid systems in which the discrete domains are affine sets and the transition maps between discrete domains are affine transformations. The simple structure of these systems results in interesting geometric properties; one of these is the notion of *spatial equivalence*. In this paper, a formal framework for describing affine hybrid systems is introduced. As an application, it is proven that every compact hybrid system $\mathbf{H}$ is spatially equivalent to a hybrid system $\mathbf{H}_{\mathrm{id}}$ in which all the transition maps are the identity. An explicit and computable construction for $\mathbf{H}_{\mathrm{id}}$ is given.

1 Introduction

This paper introduces affine hybrid systems. Affine hybrid systems are hybrid systems where the discrete domains are affine sets, and the transition maps between discrete domains are affine transformations. This definition differs from other definitions of hybrid systems that have been proposed [9], but the underlying ideas involved in the definition of affine hybrid systems have been seen in the literature [6,7]. We give a formal framework to these ideas.

Affine hybrid systems are simple, and it is this simplicity that allows us to say some useful things about them. The structure of affine hybrid systems contains a wealth of intrinsic information. Affine sets can be described in terms of matrix inequalities, and affine transformations are characterized by elements of $SE(n)$. In this paper, we use the geometric information intrinsic in affine hybrid systems to develop the idea of *spatial equivalence* between an affine hybrid system $\mathbf{H}$ and an affine hybrid system $\mathbf{G}$.

In the literature on hybrid systems, it typically is assumed that all of the transition maps of a hybrid system are the identity; all switched systems are essentially hybrid systems where the transition maps are the identity [4,10]. This assumption is very restrictive; some of the simplest hybrid systems do not satisfy this assumption, e.g., the hybrid system $\mathbb{T}^2$ constructed in Example 2.1 of this paper. For this reason, it is desirable to find a way to bridge the gap between hybrid systems where all the transition maps are the identity and hybrid systems where this is not the case.

Given an affine hybrid system $\mathbf{H}$, we would like to construct an affine hybrid system $\mathbf{H}_{\mathrm{id}}$ such that all of the transition maps are the identity. We also would like this affine hybrid system $\mathbf{H}_{\mathrm{id}}$ to be as similar to $\mathbf{H}$ as possible. In what way should these two affine hybrid systems be considered similar? Spatial equivalence

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is introduced as a way to consider an affine hybrid system $\mathbf{H}$ as similar to an affine hybrid system $\mathbf{G}$. Spatial equivalence can be thought of in an intuitive manner (see Figure 4 for a visual interpretation). Replace each edge of $\mathbf{H}$ by a sequence of edges and domains with vector fields such that if we "start" at the source of the edge, the target of the edge will be reached in some time. If the affine hybrid system obtained by appending these edges, domains and vector fields to $\mathbf{H}$ is $\mathbf{G}$, then $\mathbf{H}$ is spatially equivalent to $\mathbf{G}$. A formal definition of spatial equivalence will be given in Section 5, but having this intuitive picture in mind will be helpful.

An affine hybrid system $\mathbf{H}$ is compact if each of its domains is compact. The main theorem of this paper is:

Main Theorem. *Every compact affine hybrid system* $\mathbf{H}$ *is spatially equivalent to an affine hybrid system* $\mathbf{H}_{\mathrm{id}}$ *in which every transition map is the identity. Moreover,* $\mathbf{H}_{\mathrm{id}}$ *is computable.*

This paper begins by introducing, in Section 2, the definition of an affine hybrid system. Sections 3 and 4 present some results in affine geometry that are necessary for the proof of the Main Theorem. In Section 3, given two $(n-1)$-dimensional affine sets X and $Y = RX + p$, for $(R, p) \in SE(n)$, we determine conditions on X, R and p such that there exists an n-dimensional affine set with X and Y as faces. When these conditions are satisfied, we find a closed form solution for a set S which has X and Y as faces. This closed form solution allows us later to compute $\mathbf{H}_{\mathrm{id}}$. When there does not exist an affine set S with X and Y as faces, an admissible sequence of faces, $X = Z_0, Z_1, ..., Z_k = Y$, is introduced; it is used to construct a sequence of affine sets, $S_1, S_2, ..., S_k$, where X is a face of S_1 and Y is a face of S_k. Admissible sequences of faces are used in Section 4 to generalize the results of Section 3 by showing that if $Y = RX + p$, for $(R, p) \in SE(n)$, there is an admissible sequence of faces

$$Z_0 = X, Z_1, ..., Z_{\frac{11}{2}n(n-1)+1}, Z_{\frac{11}{2}n(n-1)+2} = Y.$$

In Section 5, the results in affine geometry that were introduced in Sections 3 and 4 are used to prove the Main Theorem and give an explicit construction for $\mathbf{H}_{\mathrm{id}}$. This is done by using the admissible sequence of faces found in Section 4 to define the domains of the hybrid system $\mathbf{H}_{\mathrm{id}}$.

2 Affine Hybrid Systems

This section introduces the notion of an affine hybrid system. An affine hybrid system consists of the following data: a set of discrete states, domains, edges and vector fields. The discrete states provide a way to index the domains. The domains are affine sets, i.e., sets that are affinely constrained. The edges provide a relationship between two faces of two domains; each edge has a source which is the face of a domain and a target which is also the face of a domain. It is required that there exists an affine transformation between the source and the target of each edge; thus each edge gives rise to a transition map, which is an affine transformation from the source of the edge to the target of the edge. The set of vector fields is a collection of vector fields that are Lipschitz on each domain.

Before we formally introduce the definition of an affine hybrid system, we will describe each of the components of its definition. This section is concluded by solidifying the concepts introduced through an example: the torus $\mathbb{T}^2$. This example also will be important later in the paper.

2.1 (Discrete states). Let $Q \subset \mathbb{Z}$ denote the set of *discrete states*. This set is finite, and the number of discrete states is given by $|Q|$. For simplicity, typically we let $Q = \{1, ..., m\}$.

2.2 (Domains). The set of *domains* is the set $D = \{D_i\}_{i \in Q}$, where each $D_i \subset \mathbb{R}^n$ is an n-dimensional affine set, i.e., a set that is affinely constrained. For each set D_i, there exists a matrix $A_i \in \mathbb{R}^{k_i \times n}$ and a vector $a_i \in \mathbb{R}^{k_i}$ such that

$$x \in D_i \qquad \Leftrightarrow \qquad A_i x + a_i \geq 0.$$

We say that D_i is *determined* by the affine constraints $A_i x + a_i$.

The boundary of D_i can be written as the union of k_i affine sets of dimension $n-1$ called the *faces* of D_i. The faces of D_i can be indexed by introducing the indexing set,

$$F_i = \{1, ..., k_i\}, \qquad i \in Q.$$

The j^{th} face of D_i is denoted by $\mathrm{Face}_j(D_i)$, where $j \in F_i$. We can pick an indexing of the faces of D_i by letting $\mathrm{Face}_j(D_i)$ be the affine set determined by the j^{th} row of A_i. More precisely, if $(A_i)_{j*}$ is the j^{th} row of A_i and $(a_i)_j$ is the j^{th} entry of a_i, then

$$x \in \mathrm{Face}_j(D_i) \qquad \Leftrightarrow \qquad \begin{pmatrix} A_i \\ -(A_i)_{j*} \end{pmatrix} x + \begin{pmatrix} a_i \\ -(a_i)_j \end{pmatrix} \geq 0. \tag{1}$$

We can define

$$A_{ij} = \begin{pmatrix} A_i \\ -(A_i)_{j*} \end{pmatrix}, \qquad a_{ij} = \begin{pmatrix} a_i \\ -(a_i)_j \end{pmatrix},$$

so $x \in \mathrm{Face}_j(D_i)$ if and only if $A_{ij}x + a_{ij} \geq 0$. Therefore, $\mathrm{Face}_j(D_i)$ is determined by the affine constraints $A_{ij}x + a_{ij}$.

2.3. For a set U with $U = \prod_{i=1}^n U_i$, denote the projections on each of the factors of U by $\pi_i : U \to U_i$.

2.4 (Edges). Define the set of *edges* as a set

$$E \subseteq \{((i,j),(k,l))\}_{(i,j) \in Q \times Q, \ \ (k,l) \in F_i \times F_j},$$

satisfying the condition that for each $e \in E$, there exists a map $T_e(x) = R_e x + p_e$, with $(R_e, p_e) \in SE(n)$, such that

$$T_e(\mathrm{Face}_{\pi_3(e)}(D_{\pi_1(e)})) = \mathrm{Face}_{\pi_4(e)}(D_{\pi_2(e)}). \tag{2}$$

In other words, an edge defines a relationship between two faces of two affine sets and an affine transformation between these faces.

More concretely, an element $e \in E$ then has the form

$$e = ((i,j),(k,l)), \qquad (i,j) \in Q \times Q, \qquad (k,l) \in F_i \times F_j,$$

so $\pi_1(e) = i$, $\pi_2(e) = j$, $\pi_3(e) = k$ and $\pi_4(e) = l$. Condition (2) allows us to write

$$T_e(\mathrm{Face}_k(D_i)) = R_e \mathrm{Face}_k(D_i) + p_e = \mathrm{Face}_l(D_j).$$

2.5. Given an edge $e \in E$, the affine transformation $T_e(x) = R_e x + p_e$ from $\mathrm{Face}_{\pi_3(e)}(D_{\pi_1(e)})$ to $\mathrm{Face}_{\pi_4(e)}(D_{\pi_2(e)})$ is called the *transition map*. The set of transition maps is the set $T = \{T_e\}_{e \in E}$.

2.6 (Vector field). A set of vector fields is a set $V = \{V_i\}_{i \in Q}$ where V_i is a Lipschitz vector field when restricted to the domain D_i. The flow of V_i on D_i is denoted by $\phi_i(t,x)$ for $x \in D_i$.

Definition 2.1. *An* affine hybrid system *is a tuple* $\mathbf{H} = (Q, D, E, V)$.

Note 2.1. From this point on, for the sake of brevity, we will refer to "affine hybrid systems" as "hybrid systems".

2.7. If for some $e \in E$, $T_e(x) = x$, then we say that the transition map associated to the edge e is the identity map. This implies that

$$\mathrm{Face}_{\pi_3(e)}(D_{\pi_1(e)}) = \mathrm{Face}_{\pi_4(e)}(D_{\pi_2(e)}).$$

A very special class of hybrid systems are hybrid systems in which every transition map is the identity, and we denote such hybrid systems as $\mathbf{H}_{\mathrm{id}}$.

Example 2.1 (The torus: $\mathbb{T}^2$). We will construct a hybrid system called the torus, which we will denote by $\mathbb{T}^2$ (see Figure 1). The torus is given by one discrete state $Q^{\mathbb{T}^2} = \{1\}$. The domain $D_1^{\mathbb{T}^2} = \{(x,y) : x \in [0,1], \quad y \in [0,1]\}$ is given by the affine constraints

$$A_1^{\mathbb{T}^2} \begin{pmatrix} x \\ y \end{pmatrix} + a_1^{\mathbb{T}^2} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

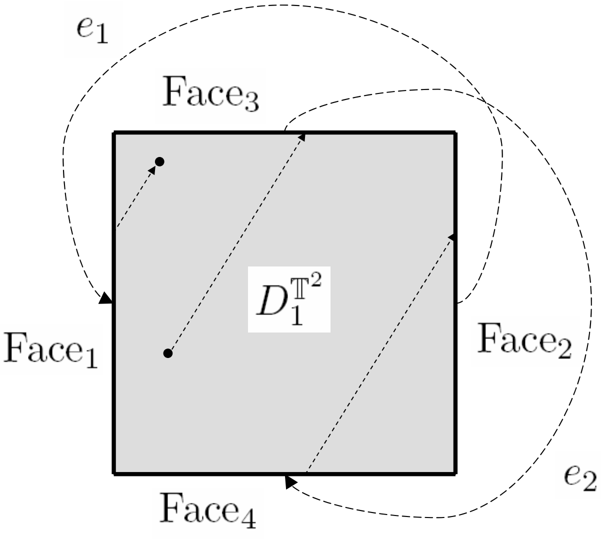


Fig. 1. The torus: $\mathbb{T}^2$.

Applying (1), the affine constraints for Face$_1(D_1^{\mathbb{T}^2})$, Face$_2(D_1^{\mathbb{T}^2})$, Face$_3(D_1^{\mathbb{T}^2})$ and Face$_4(D_1^{\mathbb{T}^2})$ are given, respectively, by

$$A_{11}^{\mathbb{T}^2}\begin{pmatrix} x \\ y \end{pmatrix} + a_{11}^{\mathbb{T}^2} = \begin{pmatrix} A_1^{\mathbb{T}^2} \\ -1 \quad 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1^{\mathbb{T}^2} \\ 0 \end{pmatrix}, \tag{3}$$

$$A_{12}^{\mathbb{T}^2}\begin{pmatrix} x \\ y \end{pmatrix} + a_{12}^{\mathbb{T}^2} = \begin{pmatrix} A_1^{\mathbb{T}^2} \\ 1 \quad 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1^{\mathbb{T}^2} \\ -1 \end{pmatrix},$$

$$A_{13}^{\mathbb{T}^2}\begin{pmatrix} x \\ y \end{pmatrix} + a_{13}^{\mathbb{T}^2} = \begin{pmatrix} A_1^{\mathbb{T}^2} \\ 0 \quad 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1^{\mathbb{T}^2} \\ -1 \end{pmatrix},$$

$$A_{14}^{\mathbb{T}^2}\begin{pmatrix} x \\ y \end{pmatrix} + a_{14}^{\mathbb{T}^2} = \begin{pmatrix} A_1^{\mathbb{T}^2} \\ 0 \quad -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1^{\mathbb{T}^2} \\ 0 \end{pmatrix}. \tag{4}$$

$E^{\mathbb{T}^2}$ consists of two edges: $e_1 = ((1,1),(2,1))$ and $e_2 = ((1,1),(3,4))$. In other words, e_1 is a relation between the top and bottom of the square and e_2 is a relation between the right and left side of the square. The associated transition maps are

$$T_{e_1}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad T_{e_2}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Finally, $V_1^{\mathbb{T}^2}$ is any vector field, Lipshitz on $D_1^{\mathbb{T}^2}$.

The advantage of defining the edges as a relationship between the faces rather than a relationship between the domains can be seen in this example. Although the expression for the edges is more complicated, the end result is a simpler definition of the hybrid system; in other references, the torus is defined with two discrete states [9].

3 Affine Sets

Given two $(n-1)$-dimensional affine sets X and $Y = RX+p$, for $(R,p) \in SE(n)$, is it possible to find an affine set S with X and Y as faces? Clearly the answer to this question is no for an arbitrary element of $SE(n)$, but it is yes if X is in the "proper position" and R and p satisfy certain assumptions. The purpose of this section is to find a closed form solution for the affine constraints defining a set S with X and Y as faces, when the assumptions on X, R and p are satisfied. This result is important because it makes the later propositions and theorem of this paper computable by way of this closed form solution. We also will use this formula repeatedly in order to compute examples, beginning with an example at the end of this section. For more detailed proofs of the results presented in this section, see [1,2].

3.1. First, recall some important facts and terminology regarding affinely constrained sets. We define a *face* of an n-dimensional affine set X, denoted by $\mathrm{Face}_i(X)$ for $i = 1, ..., k$ (where k is the number of faces), as a subset ∂X such that there exists a hyperplane H_i where $H_i \cap \partial X = \mathrm{Face}_i(X)$. This hyperplane is called the hyperplane defining $\mathrm{Face}_i(X)$. If X is determined by the affine constraints $Ax+a$, and if we define the $\mathrm{Face}_i(X)$ as the set determined by the affine constraints

$$A_i x + a_i = \begin{pmatrix} A \\ -A_{i*} \end{pmatrix} x + \begin{pmatrix} a \\ -a_i \end{pmatrix},$$

then the defining hyperplane, H_i, is given by $H_i = \{\sum_{j=1}^{n} a_{ij}x_j + a_i = 0\}$. If the smallest number of affine constraints that determine X is k, then X has k faces. Note that in this case it is always possible to define X in terms of more that k affine constraints, but never less.

Proposition 3.1. *Let X be an affine set of dimension $n-1$ in $\mathbb{R}^n$, and assume that $X \subseteq \{x_i = 0\}$. If $Y = X + p$, with $p_i \neq 0$, then there exists an affine set S such that X and Y are both faces of S. Moreover, there is a closed form solution for the affine constraints that determine S.*

Proof. If $X \subseteq \{x_i = 0\}$ is an $(n-1)$-dimensional affine set with k faces, then the affine constraints defining X can be put in the form

$$\begin{pmatrix} a_{11} & \cdots & a_{1,i-1} & 0 & a_{1,i+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{k,i-1} & 0 & a_{k,i+1} & \cdots & a_{kn} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ 0 \end{pmatrix}.$$

Since $p_i \neq 0$, if we define

$$c_k = -\frac{1}{p_i} \sum_{\substack{j = 1 \\ j \neq i}}^{n} p_j a_{kj},$$

the affine constraints for the set S are given by

$$\begin{pmatrix} a_{11} & \cdots & a_{1,i-1} & c_1 & a_{1,i+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{k,i-1} & c_k & a_{k,i+1} & \cdots & a_{kn} \\ 0 & \cdots & 0 & \mathrm{sign}(p_i) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\mathrm{sign}(p_i) & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \mathrm{sign}(p_i) p_i \end{pmatrix}. \tag{5}$$

It can be verified easily that X is the face of S given by intersecting S with the hyperplane $\{x_i = 0\}$. Similarly, Y is the face of S given by intersecting S with the hyperplane $\{x_i - p_i = 0\}$. □

3.2. Throughout this paper, we will use *angle* to refer to a scaler with values in $[-\pi, \pi)$. For $n \geq 2$, *Givens rotations* (see [5,8]) are $n \times n$ matrices of the form

$$P_{ij}(\theta) = \begin{array}{cc} \begin{array}{ccccc} & \text{column } i & & \text{column } j & \\ & \downarrow & & \downarrow & \end{array} & \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \cos\theta & & -\sin\theta & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & \sin\theta & & \cos\theta & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow \text{row } i \\ \\ \\ \leftarrow \text{row } j \\ \\ \\ \end{array} \end{array}$$

Givens rotations are important because, for every $R \in SO(n)$ with $n \geq 2$, there exists $n(n-1)/2$ angles $\theta_{ij} \in [-\pi, \pi)$ such that $R = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} P_{ij}(\theta_{ij})$ (cf. [3]). Moreover, there is a closed form solution for θ_{ij}. Therefore, understanding the effect of applying an element of $SO(n)$ to an affine set is equivalent to understanding the effect of applying a Givens rotation.

Proposition 3.2. *Let X be an affine set of dimension $n-1$ in $\mathbb{R}^n$, and assume that $X \subseteq \{x_i = 0\} \cap \{x_j \geq 0\}$. If $Y = P_{ij}(\theta) X$, with $\theta \in (0, \pi)$, then there exists an affine set S such that X and Y are both faces of S. Moreover, there is a closed form solution for the affine constraints that determine S.*

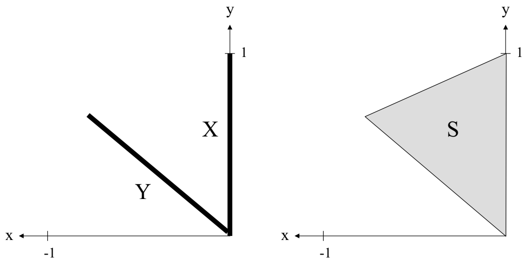


Fig. 2. Left: The sets X and Y in Example 3.1. Right: The set S with X and Y as faces.

Proof. If $X \subseteq \{x_i = 0\} \cap \{x_j \geq 0\}$ is an $(n-1)$-dimensional affine set with k faces, the affine constraints defining X can be written as

$$\begin{array}{c} \begin{array}{ccc} \text{column } i & \qquad\qquad\qquad & \text{column } j \\ \downarrow & & \downarrow \end{array} \\ \begin{pmatrix} a_{11} & \cdots & a_{1,i-1} & 0 & a_{1,i+1} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{k,i-1} & 0 & a_{k,i+1} & \cdots & a_{k,j-1} & a_{kj} & a_{k,j+1} & \cdots & a_{kn} \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ 0 \\ 0 \end{pmatrix} . \end{array} \quad (6)$$

The affine constraints for the set S with X and Y as faces are given by

$$\begin{array}{c} \begin{array}{ccc} \text{column } i & \qquad\qquad\qquad & \text{column } j \\ \downarrow & & \downarrow \end{array} \\ \begin{pmatrix} a_{11} & \cdots & a_{1,i-1} & a_{1j}(\cot\theta - \csc\theta) & a_{1,i+1} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{k,i-1} & a_{kj}(\cot\theta - \csc\theta) & a_{k,i+1} & \cdots & a_{kj} & \cdots & a_{kn} \\ 0 & \cdots & 0 & \cos\theta & 0 & \cdots & \sin\theta & \cdots & 0 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ 0 \end{pmatrix} . \end{array} \quad (7)$$

It can be verified easily that X is the face of S given by intersecting S with the hyperplane $\{x_i = 0\}$. Similarly, Y is the face of S given by intersecting S with the hyperplane $\{\cos\theta x_i + \sin\theta x_j = 0\}$. $\square$

Example 3.1. Consider the set $X = \{(x, y) : x = 0, y \in [0, 1]\}$ and $Y = P_{12}(\frac{\pi}{4})X$. Since $X \subset \{x = 0\} \cap \{y \geq 0\}$, we can apply Proposition 3.2 to determine an affine set S with X and Y as faces. The affine constraints for X are given by

$$\begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ,$$

where these affine constraints are in the same form as (6). Applying (7) gives the affine constraints for S as

$$\begin{pmatrix} (1-\sqrt{2}) & 1 \\ -(1-\sqrt{2}) & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Or this set is given by the constraints that $y \leq -(1-\sqrt{2})x + 1$, $y \geq -x$ and $x \leq 0$. The remaining constraint, that $y \geq -(1-\sqrt{2})x$, is satisfied when the other three constraints are satisfied. The set S is exactly the set that we would have hoped for (see Figure 2).

4 Admissible Sequences

For two $(n-1)$-dimensional affine sets X and $Y = RX + p$, in general it is not true that there exists an n-dimensional affine set with X and Y as faces. When there is not an affine set with X and Y as faces, the question is: does there exist a sequence of affine sets where the first affine set has X as a face, the last affine set has Y as a face, and any two adjacent affine sets in the sequence share a common face? In this section, it will be shown that for any set X and $Y = RX + p$, there exists a sequence of affine sets of this form; these sequences will be essential to the proof of the Main Theorem. The results of the previous section allow each of the affine sets in the sequence to be computed. Detailed proofs of the results of this section can be found in [1,2].

Definition 4.1. *Two $(n-1)$-dimensional affine sets, X and Y, are* admissible faces *if there exists an n-dimensional affine set $\Xi(X,Y)$ with X and Y as faces.*

4.1. If $\Xi(X,Y)$ is an affine set, for $(R,p) \in SE(n)$, there are the following properties

$$\Xi(X,Y) = \Xi(Y,X),$$
$$R\Xi(X,Y) + p = \Xi(RX+p, RY+p),$$

where $\Xi(RX+p, RY+p)$ is an affine set with $RX+p$ and $RY+p$ as faces.

4.2. We have shown in Proposition 3.1 that if $X \subset \{x_i = 0\}$ and $Y = X + p$ with $p_i \neq 0$, then X and Y are admissible faces; we can take $\Xi(X,Y)$ to be the affine set given by the affine constraints in (5). Similarly, by Proposition 3.2, if $X \subset \{x_i = 0\} \cap \{x_j \geq 0\}$ and $Y = P_{ij}(\theta)X$, for $\theta \in (0,\pi)$, then X and Y are admissible faces; we can take $\Xi(X,Y)$ to be the affine set given by the affine constraints in (7).

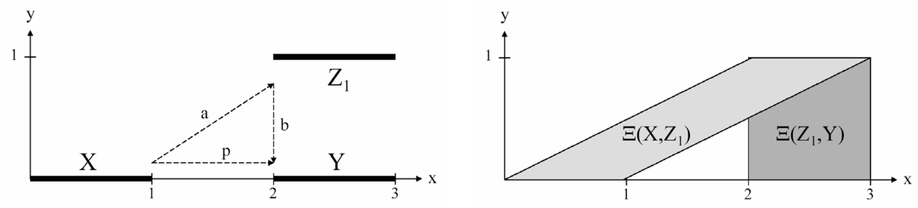


Fig. 3. Left: the sets X, Z_1 and Y in Example 4.1. Right: the affine sets $S_1 = \Xi(X, Z_1)$ and $S_2 = \Xi(Z_1, Y)$.

Definition 4.2. *A sequence $Z_0, Z_1, ..., Z_k$ of $(n-1)$-dimensional affine sets is an* admissible sequence of faces *if there exists affine sets,*

$$\Xi(Z_0, Z_1),\ \ \Xi(Z_1, Z_2),\ \ ...,\ \ \Xi(Z_{k-1}, Z_k).$$

Proposition 4.1. *Let X be an $(n-1)$-dimensional affine set and $Y = X + p$. Then there exists an admissible sequence of faces $Z_0 = X, Z_1, Z_2 = Y$.*

Proposition 4.2. *Let X be a compact $(n-1)$-dimensional affine set with $n \geq 3$, and $Y = P_{ij}(\theta)X$. Then there exists an admissible sequence of faces*

$$Z_0 = X, Z_1, ..., Z_9, Z_{11} = Y.$$

Remark 4.1. In the case where $n = 2$, an obvious modification of Proposition 4.2 gives an admissible sequence of faces $Z_0 = X$, Z_1, ..., Z_4, $Z_5 = Y$. Throughout the rest of the paper, we will assume that $n \geq 3$. All of the results are applicable to the case where $n = 2$, with the obvious modifications.

Theorem 4.1. *Let X be a compact $(n-1)$-dimensional affine set, and $Y = RX + p$, with $(R, p) \in SE(n)$. Then there exists an admissible sequence of faces*

$$Z_0 = X, Z_1, ..., Z_{\frac{11}{2}n(n-1)+1}, Z_{\frac{11}{2}n(n-1)+2} = Y.$$

Example 4.1. Let $X = \{(x, y) : x \in [0, 1],\ \ y = 0\}$, $p = (2, 0)$, and $Y = X + p$, i.e., $Y = \{(x, y) : x \in [2, 3],\ \ y = 0\}$. Therefore, X and Y are given by the affine constraints:

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \end{pmatrix},$$

respectively. It is clear that there is not a single n-dimensional affine set with X and Y as faces. This is evident in the fact that the assumptions of Proposition

3.1 are not satisfied; $X \subset \{y = 0\}$, but $p_2 = 0$. By Proposition 4.1, we can find a sequence of admissible faces $Z_0 = X, Z_1, Z_2 = Y$, and the corresponding affine sets $S_1 = \Xi(X, Z_1)$ and $S_2 = \Xi(Z_1, Y)$.

Define $a = (2, 1)$ and $b = (0, -1)$, then $a + b = p$ and $Z_1 = X + a = \{(x, y) : x \in [2, 3], \ \ y = 1\}$. We can let $S_1 = \Xi(X, Z_1)$, which is given by the affine constraints in (5). Since $Z_1 = Y - b$, $S_2 = \Xi(Z_1, Y) = \Xi(Y - b, Y) = \Xi(Y, Y - b)$. Because $Y \subset \{y = 0\}$ and $b_2 \neq 0$, applying Proposition 3.1 gives the affine constraints for S_2. Therefore, applying (5) to the affine constraints defining X and Y gives the affine constraints defining S_1 and S_2 as:

$$\begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix},$$

respectively. For a visual interpretation of these results, see Figure 3.

5 Spatial Equivalence

In this section we use the sequence of admissible faces found in Section 4 to prove the Main Theorem of this paper. In the proof of this theorem, the hybrid system $\mathbf{H}_{\mathrm{id}}$ is constructed. This allows $\mathbf{H}_{\mathrm{id}}$ to be explicitly computed, as will be seen through an example following the proof of the Main Theorem.

5.1. A hybrid system $\mathbf{H} = (Q^{\mathbf{H}}, D^{\mathbf{H}}, E^{\mathbf{H}}, V^{\mathbf{H}})$ is said to be *spatially equivalent* to a hybrid system $\mathbf{G} = (Q^{\mathbf{G}}, D^{\mathbf{G}}, E^{\mathbf{G}}, V^{\mathbf{G}})$ if the following conditions hold:

1. $|Q^{\mathbf{G}}| \geq |Q^{\mathbf{H}}| = m$.
2. For every $i \leq m$, $D_i^{\mathbf{H}} = D_i^{\mathbf{G}}$ and for $i > m$, there exist admissible faces X_i and Y_i such that $D_i^{\mathbf{G}} = \Xi(X_i, Y_i)$.
3. For every edge $e \in E^{\mathbf{H}}$ there exists a sequence of k discrete states $\nu(1)$, ..., $\nu(k) > m$ and edges $\eta_1, ..., \eta_{k+1} \in E^{\mathbf{G}}$ such that

$$\begin{aligned} T_{\eta_1}(\mathrm{Face}_{\pi_3(e)}(D^{\mathbf{H}}_{\pi_1(e)})) &= X_{\nu(1)}, \\ T_{\eta_2}(Y_{\nu(1)}) &= X_{\nu(2)}, \\ &\vdots \\ T_{\eta_{k+1}}(Y_{\nu(k)}) &= \mathrm{Face}_{\pi_4(e)}(D^{\mathbf{H}}_{\pi_2(e)}). \end{aligned}$$

   In the special case where $k = 0$, we require η_1 to be an edge such that

$$T_{\eta_1}(\mathrm{Face}_{\pi_3(e)}(D^{\mathbf{H}}_{\pi_1(e)})) = \mathrm{Face}_{\pi_4(e)}(D^{\mathbf{H}}_{\pi_2(e)}).$$

4. For $i \leq m$, $V_i^{\mathbf{H}} = V_i^{\mathbf{G}}$, and for $i > m$, $V_i^{\mathbf{G}}$ is a vector field such that $\phi_i^{\mathbf{G}}(1, X_i) = Y_i$, where $\phi_i^{\mathbf{G}}(t, x)$ is the solution to $V_i^{\mathbf{G}}$.

Remark 5.1. Note that spatial equivalence is not an equivalence relation. The term "equivalence" is used in order to stress the equivalence of the qualitative behavior of $\mathbf{H}$ and $\mathbf{G}$, when $\mathbf{H}$ is spatially equivalent to $\mathbf{G}$. Although it might be appropriate to replace "spatial equivalence" by a term such as "spatial embedding", the authors are concerned that this term would not stress the behavioral similarities of the two hybrid systems.

Definition 5.1. *A hybrid system* $\mathbf{H}$ *is* compact *if each of its domains is compact.*

Theorem 5.1. *If* $\mathbf{H}$ *is a compact hybrid system, then* $\mathbf{H}$ *is spatially equivalent to a hybrid system* $\mathbf{H}_{\mathrm{id}}$ *such that every transition map is the identity, i.e.,* $T_\eta = \mathrm{id}$ *for every* $\eta \in E^{\mathbf{H}_{\mathrm{id}}}$*. Moreover,* $\mathbf{H}_{\mathrm{id}}$ *is computable.*

Proof. In order to prove this theorem, we will explicitly construct the hybrid system $\mathbf{H}_{\mathrm{id}}$. First, we define

$$E^{\mathbf{H}}_{\not\mathrm{id}} := \{e \in E^{\mathbf{H}} : \quad T_e \neq \mathrm{id}\},$$

which is the set of edges such that the associated transition map is not the identity. If $|E^{\mathbf{H}}_{\not\mathrm{id}}| = k$, then we can write the elements of $E^{\mathbf{H}}_{\not\mathrm{id}}$ as $e_1, ..., e_k$ (by arbitrarily indexing them). For simplicity of notation, define the functions

$$\begin{aligned} f(n) &= \frac{11}{2} n(n-1) + 2, \\ g(m,n,i) &= (i-1)f(n) + m + 1, \end{aligned}$$

which will be used throughout the course of the construction.

Construction of $\mathbf{H}_{\mathrm{id}}$

$\underline{Q^{\mathbf{H}_{\mathrm{id}}}}$: If $Q^{\mathbf{H}} = \{1, ..., m\}$, then define $Q^{\mathbf{H}_{\mathrm{id}}} = \{1, ..., m + kf(n)\}$, with $k = |E^{\mathbf{H}}_{\not\mathrm{id}}|$.

$\underline{D^{\mathbf{H}_{\mathrm{id}}}}$: For $i \leq m$, define $D_i^{\mathbf{H}_{\mathrm{id}}} = D_i^{\mathbf{H}}$. If $A_i x + a_i$ are the affine constraints determining $D_i^{\mathbf{H}}$, then we also let $A_i x + a_i$ (with the order of the rows maintained) be the affine constraints determining $D_i^{\mathbf{H}_{\mathrm{id}}}$. In particular, this implies that $\mathrm{Face}_j(D_i^{\mathbf{H}_{\mathrm{id}}}) = \mathrm{Face}_j(D_i^{\mathbf{H}})$.

Now we can construct the other domains of $\mathbf{H}_{\mathrm{id}}$. For every edge $e_i \in E^{\mathbf{H}}_{\not\mathrm{id}}$, $1 \leq i \leq k$, the transition map is given by $T_{e_i}(x) = R_{e_i} x + p_{e_i}$, or we have

$$\mathrm{Face}_{\pi_4(e_i)}(D^{\mathbf{H}}_{\pi_2(e_i)}) = R_{e_i} \mathrm{Face}_{\pi_3(e_i)}(D^{\mathbf{H}}_{\pi_1(e_i)}) + p_{e_i}.$$

Now by Theorem 4.1 we have the following admissible sequence of faces

$$Z_0^i = \mathrm{Face}_{\pi_3(e_i)}(D^{\mathbf{H}}_{\pi_1(e_i)}), \ Z_1^i, \ ..., \ Z^i_{f(n)-1}, \ Z^i_{f(n)} = \mathrm{Face}_{\pi_4(e_i)}(D^{\mathbf{H}}_{\pi_2(e_i)}).$$

Setting

$$\begin{aligned} X_{g(m,n,i)} &= Z_0^i, \\ X_{g(m,n,i)+1} &= Z_1^i &= Y_{g(m,n,i)}, \\ &\vdots \\ X_{g(m,n,i)+f(n)-1} &= Z_{f(n)-1}^i &= Y_{g(m,n,i)+f(n)-2}, \\ & Z_{f(n)}^i &= Y_{g(m,n,i)+f(n)-1}, \end{aligned}$$

define the domains

$$D^{\mathbf{H}_{\mathrm{id}}}_{g(m,n,i)+j} = \Xi(X_{g(m,n,i)+j}, Y_{g(m,n,i)+j}), \qquad 1 \leq i \leq k, \qquad 0 \leq j \leq f(n)-1.$$

It can be verified that for these values of i and j, $g(m,n,i)+j$ takes all values from $m+1$ to $m+kf(n)$, inclusive, and with no repeats.

$\underline{E^{\mathbf{H}_{\mathrm{id}}}}$: If $e \in E^{\mathbf{H}}$ and $e \notin E^{\mathbf{H}}_{\not{\mathrm{id}}}$, then the associated transition map is $T_e = \mathrm{id}$. So define an edge $\eta(e) \in E^{\mathbf{H}_{\mathrm{id}}}$ to be $\eta(e) = e$. It follows that $T_{\eta(e)} = \mathrm{id}$. If $e \in E^{\mathbf{H}}_{\not{\mathrm{id}}}$, then $e = e_i$ for $i \in \{1, ..., k\}$. We can now define a set of edges $\eta_1(e_i), \eta_2(e_i), ..., \eta_{f(n)+1}(e_i) \in E^{\mathbf{H}_{\mathrm{id}}}$ as follows: if we index the faces of $D^{\mathbf{H}_{\mathrm{id}}}_{g(m,n,i)+j}$ such that

$$X_{g(m,n,i)+j} = \mathrm{Face}_1(D^{\mathbf{H}_{\mathrm{id}}}_{g(m,n,i)+j}), \qquad Y_{g(m,n,i)+j} = \mathrm{Face}_2(D^{\mathbf{H}_{\mathrm{id}}}_{g(m,n,i)+j}),$$

then we define

$$\begin{aligned} \eta_1(e_i) &= ((\pi_1(e_i), g(m,n,i)), (\pi_3(e_i), 1), \\ \eta_2(e_i) &= ((g(m,n,i), g(m,n,i)+1), (2,1)), \\ &\vdots \\ \eta_j(e_i) &= ((g(m,n,i)+j-2, g(m,n,i)+j-1), (2,1)), \\ &\vdots \\ \eta_{f(n)}(e_i) &= ((g(m,n,i)+f(n)-2, g(m,n,i)+f(n)-1), (2,1)), \\ \eta_{f(n)+1}(e_i) &= ((g(m,n,i)+f(n)-1, \pi_2(e_i)), (2, \pi_4(e_i)). \end{aligned}$$

The associated transition maps are

$$\begin{aligned} T_{\eta_1(e_i)} &: \mathrm{Face}_{\pi_3(e_i)}(D^{\mathbf{H}}_{\pi_1(e_i)}) \to X_{g(m,n,i)}, \\ T_{\eta_j(e_i)} &: Y_{g(m,n,i)+j-2} \to X_{g(m,n,i)+j-1}, \qquad 1 < j \leq f(n), \\ T_{\eta_{f(n)+1}(e_i)} &: Y_{g(m,n,i)+f(n)-1} \to \mathrm{Face}_{\pi_4(e_i)}(D^{\mathbf{H}}_{\pi_2(e_i)}), \end{aligned}$$

so

$$\begin{aligned} \mathrm{Face}_{\pi_3(e_i)}(D^{\mathbf{H}}_{\pi_1(e_i)}) &= X_{g(m,n,i)}, \\ Y_{g(m,n,i)+j-2} &= X_{g(m,n,i)+j-1}, \qquad 1 < j \leq f(n), \\ Y_{g(m,n,i)+f(n)-1} &= \mathrm{Face}_{\pi_4(e_i)}(D^{\mathbf{H}}_{\pi_2(e_i)}). \end{aligned}$$

By definition, it follows that

$$T_{\eta_1(e_i)} = T_{\eta_2(e_i)} = \cdots = T_{\eta_{f(n)+1}(e_i)} = \mathrm{id}.$$

If we apply this construction to every edge in $E^{\mathbf{H}}$ the result is the set $E^{\mathbf{H}_{\mathrm{id}}}$. It is clear that for every edge in $\eta \in E^{\mathbf{H}_{\mathrm{id}}}$, $T_\eta = \mathrm{id}$. It also follows that $|E^{\mathbf{H}_{\mathrm{id}}}| = |E^{\mathbf{H}}| + 2f(n)$.

$\underline{V^{\mathbf{H}_{\mathrm{id}}}}$: If $i \leq m$, define $V_i^{\mathbf{H}_{\mathrm{id}}} = V_i^{\mathbf{H}}$. If $i > m$, then $D_i^{\mathbf{H}_{\mathrm{id}}} = \Xi(X_i, Y_i)$, where X_i and Y_i differ by an element of $SE(n)$, i.e., $Y_i = Q_i X_i + q_i$. Using this, we define

$$\phi_i^{\mathbf{H}_{\mathrm{id}}}(t,x) = (1-t)x + t(Q_i x + q_i), \qquad V_i^{\mathbf{H}_{\mathrm{id}}}(x) = \frac{d}{dt}(\phi_i^{\mathbf{H}_{\mathrm{id}}}(t,x)),$$

and we have the property that $\phi_i^{\mathbf{H}_{\mathrm{id}}}(1, X_i) = Q_i X_i + q_i = Y_i$.

To conclude the proof we note that in the process of constructing $\mathbf{H}_{\mathrm{id}}$ we have shown that $\mathbf{H}$ and $\mathbf{H}_{\mathrm{id}}$ satisfy properties 1-4 of Paragraph 5.1, hence $\mathbf{H}$ is spatially equivalent to $\mathbf{H}_{\mathrm{id}}$. $\square$

Remark 5.2. Note that the hybrid system we constructed in the proof of Theorem 5.1 is not unique. Moreover, the number of discrete states given in the construction is not necessarily the smallest number of discrete states needed to construct a spatially equivalent hybrid system. For example, if for every edge $e \in E^{\mathbf{H}}$, the faces $\mathrm{Face}_{\pi_3(e)}(D^{\mathbf{H}}_{\pi_1(e)})$ and $\mathrm{Face}_{\pi_4(e)}(D^{\mathbf{H}}_{\pi_2(e)})$ are admissible (see Definition 4.1), then we can construct a hybrid system $\mathbf{H}_{\mathrm{id}}$, spatially equivalent to $\mathbf{H}$, with $Q^{\mathbf{H}_{\mathrm{id}}} = \{1, ..., m+k\}$. This will be the case in the following example.

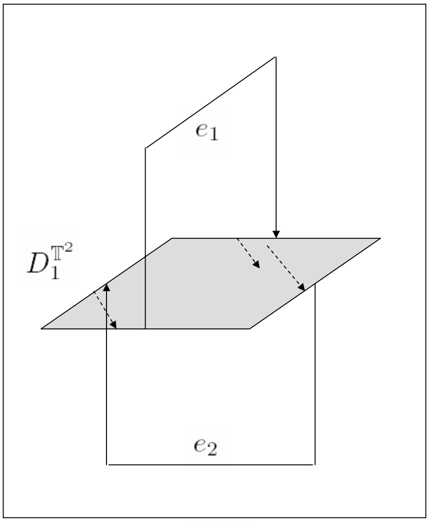
Example 5.1. We will construct $\mathbb{T}^2_{\mathrm{id}}$, or a hybrid system spatially equivalent to $\mathbb{T}^2$ (see Example 2.1) where every transition map is the identity. In this case we have two edges, $e_1, e_2 \in E^{\mathbb{T}^2}$, and $|E^{\mathbb{T}^2}_{\mathrm{id}}| = 2$. First, note that we can define $\mathbb{T}^2_{\mathrm{id}}$ in terms of fewer than the number of discrete states given in the proof of Theorem 5.1 because $\mathrm{Face}_{\pi_3(e_i)}(D^{\mathbf{H}}_{\pi_1(e_i)})$ and $\mathrm{Face}_{\pi_4(e_i)}(D^{\mathbf{H}}_{\pi_2(e_i)})$ are admissible faces for $i = 1, 2$ (see Remark 5.2).

Set $Q^{\mathbb{T}^2_{\mathrm{id}}} = \{1, 2, 3\}$, and let $D_1^{\mathbb{T}^2_{\mathrm{id}}} = D_1^{\mathbb{T}^2}$, which we defined in Example 2.1. To construct $D_2^{\mathbb{T}^2_{\mathrm{id}}}$ and $D_3^{\mathbb{T}^2_{\mathrm{id}}}$, note that

$$\mathrm{Face}_2(D_1^{\mathbb{T}^2}) = \mathrm{Face}_1(D_1^{\mathbb{T}^2}) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathrm{Face}_3(D_1^{\mathbb{T}^2}) = \mathrm{Face}_4(D_1^{\mathbb{T}^2}) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\mathrm{Face}_1(D_1^{\mathbb{T}^2}) \subset \{x = 0\}$ and $\mathrm{Face}_4(D_1^{\mathbb{T}^2}) \subset \{y = 0\}$, by applying Proposition 3.1 to the affine constraints $A_{11}^{\mathbb{T}^2} x + a_{11}^{\mathbb{T}^2}$ and $A_{14}^{\mathbb{T}^2} x + a_{14}^{\mathbb{T}^2}$, given in equations (3) and (4), it can be verified that

$$\Xi(\mathrm{Face}_1(D_1^{\mathbb{T}^2}), \mathrm{Face}_2(D_1^{\mathbb{T}^2})) = \Xi(\mathrm{Face}_4(D_1^{\mathbb{T}^2}), \mathrm{Face}_3(D_1^{\mathbb{T}^2})) = D_1^{\mathbb{T}^2}.$$



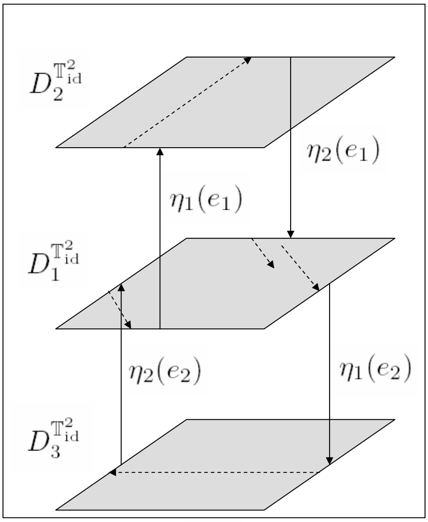


Fig. 4. Left: $\mathbb{T}^2$. Right: $\mathbb{T}^2_{\mathrm{id}}$.

Now as in the proof of Theorem 5.1, define

$$\begin{aligned} X_2 &= \mathrm{Face}_2(D_1^{\mathbb{T}^2}), & Y_2 &= \mathrm{Face}_1(D_1^{\mathbb{T}^2}), \\ X_3 &= \mathrm{Face}_3(D_1^{\mathbb{T}^2}), & Y_3 &= \mathrm{Face}_4(D_1^{\mathbb{T}^2}), \end{aligned}$$

then

$$D_2^{\mathbb{T}^2_{\mathrm{id}}} = \Xi(X_2, Y_2) = D_1^{\mathbb{T}^2}, \qquad D_3^{\mathbb{T}^2_{\mathrm{id}}} = \Xi(X_3, Y_3) = D_1^{\mathbb{T}^2},$$

and $D^{\mathbb{T}^2_{\mathrm{id}}} = \{D_1^{\mathbb{T}^2_{\mathrm{id}}}, D_2^{\mathbb{T}^2_{\mathrm{id}}}, D_3^{\mathbb{T}^2_{\mathrm{id}}}\}$.

Now we will determine the edges of $\mathbb{T}^2_{\mathrm{id}}$. As in the proof of Theorem 5.1, index the faces of $D_2^{\mathbb{T}^2_{\mathrm{id}}}$ and $D_3^{\mathbb{T}^2_{\mathrm{id}}}$ such that

$$\begin{aligned} X_2 &= \mathrm{Face}_1(D_2^{\mathbb{T}^2_{\mathrm{id}}}), & Y_2 &= \mathrm{Face}_2(D_2^{\mathbb{T}^2_{\mathrm{id}}}), \\ X_3 &= \mathrm{Face}_1(D_3^{\mathbb{T}^2_{\mathrm{id}}}), & Y_3 &= \mathrm{Face}_2(D_3^{\mathbb{T}^2_{\mathrm{id}}}), \end{aligned}$$

and for the two edges $e_1, e_2 \in E^{\mathbb{T}^2}$, define

$$\begin{aligned} \eta_1(e_1) &= ((1,2),(2,1)), & \eta_1(e_2) &= ((1,3),(4,1)), \\ \eta_2(e_1) &= ((2,1),(2,1)), & \eta_2(e_2) &= ((3,1),(2,3)). \end{aligned}$$

Set $E^{\mathbb{T}^2_{\mathrm{id}}} = \{\eta_1(e_1), \eta_2(e_1), \eta_1(e_2), \eta_2(e_2)\}$, and note that the corresponding transition maps $T_{\eta_1(e_1)} = T_{\eta_2(e_1)} = T_{\eta_1(e_2)} = T_{\eta_2(e_2)} = \mathrm{id}$ (see Figure 4).

Finally, define $V_1^{\mathbb{T}^2_{\mathrm{id}}} = V_1^{\mathbb{T}^2}$ and

$$V_2^{\mathbb{T}^2_{\mathrm{id}}}(x) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad V_3^{\mathbb{T}^2_{\mathrm{id}}}(x) = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

So, $V^{\mathbb{T}^2_{\mathrm{id}}} = \{V_1^{\mathbb{T}^2_{\mathrm{id}}}, V_2^{\mathbb{T}^2_{\mathrm{id}}}, V_3^{\mathbb{T}^2_{\mathrm{id}}}\}$. This completes the construction of $\mathbb{T}^2_{\mathrm{id}}$.

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