Hybrid Model Structures

by

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Abstract

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Hybrid systems are systems that display both continuous and discrete behavior, and so have important applications to engineering, e.g., mechanical systems undergoing impacts are naturally modeled by systems of this form. As with dynamical systems, understanding the homotopy-theoretic properties of hybrid systems—including topology and homology—allows for important insights into the behavior of these systems. Unlike dynamical systems, there is currently not a mathematical framework in which to understand hybrid systems homotopically. This dissertation provides the first steps toward establishing such a framework.

Fundamental to our investigations is the theory of model categories, which provides a method for "doing homotopy theory" on general categories which satisfy certain axioms. Originally formulated by Quillen [21] in 1967, model category theory has since blossomed into a full-fledged area of research capable of addressing homotopy-theoretic questions in a general context. Some of the quintessential model categories are the category of topological spaces, the category of simplicial sets and the category of chain complexes—the model structure of these categories plays a fundamental role in algebraic topology and homology. Therefore, understanding hybrid systems in the context of model categories will allow one to understand the homotopy-theoretic properties of these systems, laying the ground work for *hybrid homotopy theory*.

The core observation of this thesis is that hybrid systems, and more generally *hybrid objects*, can be represented categorically as diagrams over a category. That is, given a category M consisting of the non-hybrid objects of interest, e.g., topological spaces, a hybrid object over this category consists of a small category, $\mathscr{D}$, that captures the discrete structure of the hybrid object together with a functor:

$$\mathbf{A} : \mathscr{D} \to \mathsf{M},$$

that captures the continuous structure of this hybrid object. Therefore, given a category M, we are interested in studying the functor category $\mathsf{M}^{\mathscr{D}}$ and more generally the category of hybrid objects over M, $\mathsf{Hy}(\mathsf{M})$.

Before delving into the relationship between hybrid objects and model structures, we lay the groundwork for these concepts by considering adjunctions; these are fundamental in the study of model categories. The goal is to understand the role that adjunctions play in a hybrid setting. The study of adjunctions is first motivated by the study of limits; it is demonstrated that if a category M is complete, i.e., limits exist, then $\mathsf{Hy}(\mathsf{M})$ is complete. Adjunctions are then formally introduced. With this concept in mind, colimits are introduced; it is demonstrated that if M is cocomplete, i.e., colimits exist, then $\mathsf{Hy}(\mathsf{M})$ is cocomplete. Having studied adjunctions in a "non-hybrid" setting, it is shown how an adjunction between categories extends to an adjunction between categories of hybrid objects.

The final portion of this thesis is devoted to understanding hybrid model structures, i.e., the model structure of diagrams in a model category. Preexisting mathematical constructions can be used to demonstrate that if M is a model category, then so is $\mathsf{M}^{\mathscr{D}}$; the model structure on $\mathsf{M}^{\mathscr{D}}$ is induced by the model structure on M. Therefore, we can understand "hybrid homotopy theory" in terms of its "non-hybrid" counterpart. This connection is firmly established through the use of homotopy colimits, i.e., one can define a *cofibrantly homotopy meaningful* model structure on $\mathsf{M}^{\mathscr{D}}$, and the homotopy colimit is the total left derived functor of the colimit.

Professor Mariusz Wodzicki
Dissertation Committee Chair

To my sister,

Nora C. Ames.

Contents

Chapter 1

Hybrid Objects

The starting point for introducing the notion of a hybrid object over a category is the observation that systems that display both continuous and discrete behavior, i.e., hybrid systems, can be represented by a D-category, $\mathscr{D}$, together with a functor:

$$\mathbf{A} : \mathscr{D} \to \mathsf{C},$$

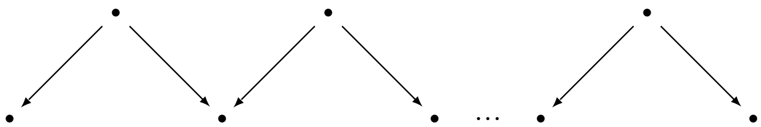
where C is an arbitrary category. This relates hybrid systems to the two most fundamental objects in category theory: a functor and a natural transformation. More generally, it allows one to use preexisting mathematical constructions, such as the one that defines a homotopy meaningful model structure on diagrams in a model category, in the study of hybrid systems.

The concepts introduced in this chapter follow from [2]; here we only introduce the constructions necessary to the later chapters.

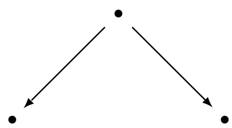
1.1 D-categories

The notion that will be needed in order to introduce a hybrid object over a category is a D-category, where here “D” is used in order to illustrate that these categories represent the discrete structure of a hybrid object.

A D-category is a small category in which every diagram has the form:



That is, a D-category has as its basic atomic unit a diagram of the form:



and any other diagram in this category must be obtainable by gluing such atomic units along the target of a morphism (and not the source). More formally, consider the following:

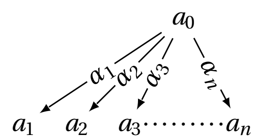
Definition 1.1. A *D-category* is a small category $\mathscr{D}$ satisfying the following two axioms:

AD1 Every object in $\mathscr{D}$ is either the domain of a non-identity morphism in $\mathscr{D}$ or the codomain of a non-identity morphism but never both, i.e., for every diagram

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n$$

in $\mathscr{D}$, all but one morphism must be the identity (the longest chain of composable non-identity morphisms is of length one).

AD2 If an object in $\mathscr{D}$ is the domain of a non-identity morphism, then it is the domain of exactly two non-identity morphisms, i.e., for every diagram in $\mathscr{D}$ of the form



consisting of all morphisms with domain a_0, either all of the morphisms are the identity or two and only two morphisms are not the identity.

1.1.1 Important objects in D-categories.

Let $\mathscr{D}$ be a D-category. We use $\mathsf{Mor}(\mathscr{D})$ to denote the morphisms of $\mathscr{D}$, i.e.,

$$\mathsf{Mor}(\mathscr{D}) = \bigcup_{(a,b)\in \mathsf{Ob}(\mathscr{D})\times \mathsf{Ob}(\mathscr{D})} \mathsf{Hom}_{\mathscr{D}}(a,b),$$

and $\mathsf{Mor}_{\not{\mathrm{id}}}(\mathscr{D})$ to denote the set of non-identity morphisms of $\mathscr{D}$, i.e.,

$$\mathsf{Mor}_{\not{\mathrm{id}}}(\mathscr{D}) = \{\alpha \in \mathsf{Mor}(\mathscr{D}) : \alpha \neq \mathrm{id}\}.$$

For a morphism $\alpha : a \to b$ in $\mathscr{D}$, its domain (or source) is denoted by $\mathrm{dom}(\alpha) = a$ and its codomain (or target) is denoted by $\mathrm{cod}(\alpha) = b$.

For D-categories, there are two sets of objects that are of particular interest; these are subsets of $\mathrm{Ob}(\mathscr{D})$. The first of these is termed the *edge set of* $\mathscr{D}$, denoted by $\mathsf{E}(\mathscr{D})$, and defined to be:

$$\mathsf{E}(\mathscr{D}) = \{a \in \mathrm{Ob}(\mathscr{D}) : \ a = \mathrm{dom}(\alpha),\ a = \mathrm{dom}(\beta),\ \alpha, \beta \in \mathrm{Mor}_{\not\mathrm{id}}(\mathscr{D}),\ \alpha \neq \beta\}.$$

That is, for all $a \in \mathsf{E}(\mathscr{D})$ there are two and only two morphisms (which are not the identity) $\alpha, \beta \in \mathrm{Mor}(\mathscr{D})$ such that $a = \mathrm{dom}(\alpha)$ and $a = \mathrm{dom}(\beta)$, so we denote these morphisms by s_a and t_a, respectively. Conversely, given a morphism $\gamma \in \mathrm{Mor}_{\not\mathrm{id}}(\mathscr{D})$, there exists a unique $a \in \mathsf{E}(\mathscr{D})$ such that $\gamma = \mathsf{s}_a$ or $\gamma = \mathsf{t}_a$. Therefore, every object $a \in \mathsf{E}(\mathscr{D})$ sits in a diagram of the form:

$$\begin{array}{ccc} & \mathrm{dom}(\mathsf{s}_a) = a = \mathrm{dom}(\mathsf{t}_a) & \\ {}^{\mathsf{s}_a}\swarrow & & \searrow^{\mathsf{t}_a} \\ b = \mathrm{cod}(\mathsf{s}_a) & & \mathrm{cod}(\mathsf{t}_a) = c \end{array} \qquad (1.1)$$

Note that giving all diagrams of this form (for which there is one for each $a \in \mathsf{E}(\mathscr{D})$) gives all the objects in $\mathscr{D}$, i.e., every object of $\mathscr{D}$ is the target of s_a or t_a, or their source, for some $a \in \mathsf{E}(\mathscr{D})$.

Define the *vertex set of* $\mathscr{D}$ by:

$$\mathsf{V}(\mathscr{D}) = (\mathsf{E}(\mathscr{D}))^c,$$

where here $(\mathsf{E}(\mathscr{D}))^c$ is the complement of $\mathsf{E}(\mathscr{D})$ in the set $\mathrm{Ob}(\mathscr{D})$. It follows by definition that

$$\begin{aligned} \mathsf{E}(\mathscr{D}) \cap \mathsf{V}(\mathscr{D}) &= \emptyset, \\ \mathsf{E}(\mathscr{D}) \cup \mathsf{V}(\mathscr{D}) &= \mathrm{Ob}(\mathscr{D}). \end{aligned}$$

The above choice of morphisms s_a and t_a can be used to define an orientation on a D-category. From this point on, we will only consider oriented D-categories. Therefore, we introduce the following:

Definition 1.2. A *D-category* is a small category $\mathscr{D}$ such that:

- There exist two subsets of $\mathrm{Ob}(\mathscr{D})$, $\mathsf{E}(\mathscr{D})$ and $\mathsf{V}(\mathscr{D})$, termed the *edge set* and *vertex set*, satisfying:

$$\begin{aligned} \mathsf{E}(\mathscr{D}) \cap \mathsf{V}(\mathscr{D}) &= \emptyset, \\ \mathsf{E}(\mathscr{D}) \cup \mathsf{V}(\mathscr{D}) &= \mathrm{Ob}(\mathscr{D}), \end{aligned}$$

- There exists a pair of functions:

$$\mathsf{E}(\mathscr{D}) \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} \mathrm{Mor}_{\not\mathrm{id}}(\mathscr{D}),$$

  such that:

$$\begin{aligned} \mathsf{s}(\mathsf{E}(\mathscr{D})) \cap \mathsf{t}(\mathsf{E}(\mathscr{D})) &= \emptyset, \\ \mathsf{s}(\mathsf{E}(\mathscr{D})) \cup \mathsf{t}(\mathsf{E}(\mathscr{D})) &= \mathrm{Mor}_{\not\mathrm{id}}(\mathscr{D}). \end{aligned}$$

  The pair (s, t) is termed an *orientation* of $\mathscr{D}$.

- The following diagram:

$$\begin{array}{ccc} & & \mathsf{E}(\mathscr{D}) \\ & \overset{\mathsf{id}}{\nearrow} & \uparrow \mathsf{dom} \\ \mathsf{E}(\mathscr{D}) & \underset{\mathsf{t}}{\overset{\mathsf{s}}{\rightrightarrows}} & \mathsf{Mor}_{\not\mathsf{id}}(\mathscr{D}) \\ & & \downarrow \mathsf{cod} \\ & & \mathsf{V}(\mathscr{D}) \end{array} \tag{1.2}$$

commutes.

In particular, for a D-category $\mathscr{D}$, the definition of such a category implies that for every $a \in \mathsf{E}(\mathscr{D})$, there is a diagram of the form (1.1). To verify that the (oriented) D-categories, as defined in 1.2, satisfy the axioms of a D-category as given in Definition 1.1, we state the following:.

Lemma 1.1. *A D-category, as defined in 1.2, satisfies* **AD1** *and* **AD2**.

D-categories, as introduced in Definition 1.2, are *oriented* since a specific labeling of the morphisms was chosen. From this point on, all D-categories will be oriented.

1.1.2 The category of D-categories.

Define the category of D-categories, Dcat, to have objects D-categories. A morphism between two D-categories, $\mathscr{D}$ and $\mathscr{D}'$ (with functions (s,t) and $(\mathsf{s}',\mathsf{t}')$, respectively), is a functor $\vec{F}:\mathscr{D}\to\mathscr{D}'$ such that

$$\vec{F}(\mathsf{E}(\mathscr{D})) \subseteq \mathsf{E}(\mathscr{D}'), \qquad \vec{F}(\mathsf{V}(\mathscr{D})) \subseteq \mathsf{V}(\mathscr{D}'), \tag{1.3}$$

and the following diagrams

$$\begin{array}{ccc} \mathsf{E}(\mathscr{D}) & \xrightarrow{\vec{F}} & \mathsf{E}(\mathscr{D}') \\ \downarrow \mathsf{s} & & \downarrow \mathsf{s}' \\ \mathsf{Mor}_{\not\mathsf{id}}(\mathscr{D}) & \xrightarrow{\vec{F}} & \mathsf{Mor}_{\not\mathsf{id}}(\mathscr{D}') \end{array} \qquad \begin{array}{ccc} \mathsf{E}(\mathscr{D}) & \xrightarrow{\vec{F}} & \mathsf{E}(\mathscr{D}') \\ \downarrow \mathsf{t} & & \downarrow \mathsf{t}' \\ \mathsf{Mor}_{\not\mathsf{id}}(\mathscr{D}) & \xrightarrow{\vec{F}} & \mathsf{Mor}_{\not\mathsf{id}}(\mathscr{D}') \end{array} \tag{1.4}$$

commute. By requiring these diagrams to commute, it implies that for all diagrams of the form:

$$\begin{array}{ccc} & a & \\ {}^{\mathsf{s}_a}\swarrow & & \searrow^{\mathsf{t}_a} \\ b & & c \end{array}$$

in $\mathscr{D}$, i.e., $a \in \mathsf{E}(\mathscr{D})$ and $b, c \in \mathsf{V}(\mathscr{D})$, there are corresponding diagrams:

$$\begin{array}{ccc} & \vec{F}(a) & \\ {}^{\vec{F}(\mathsf{s}_a)=\mathsf{s}'_{\vec{F}(a)}}\swarrow & & \searrow^{\vec{F}(\mathsf{t}_a)=\mathsf{t}'_{\vec{F}(a)}} \\ \vec{F}(b) & & \vec{F}(c) \end{array}$$

in $\mathscr{D}'$, where $\vec{F}(a) \in \mathsf{E}(\mathscr{D}')$ and $\vec{F}(b), \vec{F}(c) \in \mathsf{V}(\mathscr{D}')$.

1.1.3 D-categories and graphs. Recall that a (*directed* or *oriented*) graph is a pair $\Gamma = (Q, E)$, where Q is a set of vertices and E is a set of edges, together with a pair of functions:

$$E \underset{\mathsf{tar}}{\overset{\mathsf{sor}}{\rightrightarrows}} Q$$

called the source and target functions; for $e \in E$, $\mathsf{sor}(e)$ is the source of e and $\mathsf{tar}(e)$ is the target of e. A morphisms of graphs consists of a pair $F = (F_Q, F_E) : \Gamma = (Q, E) \to \Gamma' = (Q', E')$, where $F_Q : Q \to Q'$ and $F_E : E \to E'$, such that the following diagrams commute:

$$\begin{array}{ccc} E & \xrightarrow{F_E} & E' \\ {\scriptstyle \mathsf{sor}}\downarrow & & \downarrow{\scriptstyle \mathsf{sor}'} \\ Q & \xrightarrow{F_Q} & Q' \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{F_E} & E' \\ {\scriptstyle \mathsf{tar}}\downarrow & & \downarrow{\scriptstyle \mathsf{tar}'} \\ Q & \xrightarrow{F_Q} & Q' \end{array} \tag{1.5}$$

Thus we have defined the category of graphs, Grph.

D-categories can be essentially thought of as graphs. (Although, in the context of hybrid systems, it is not sufficient to work with graphs.) Specifically, one can associate to a D-category $\mathscr{D}$ a graph

$$\Gamma_{\mathscr{D}} = (Q_{\mathscr{D}}, E_{\mathscr{D}}) := (\mathsf{V}(\mathscr{D}), \mathsf{E}(\mathscr{D})),$$

with source and target maps given by:

$$E_{\mathscr{D}} \underset{\mathsf{tar} = \mathsf{cod}(\mathsf{t}_{(-)})}{\overset{\mathsf{sor} = \mathsf{cod}(\mathsf{s}_{(-)})}{\rightrightarrows}} Q_{\mathscr{D}}.$$

Conversely, given a graph Γ, one associates to this graph a D-category $\mathscr{D}_\Gamma$ by defining the objects of this category to be:

$$\mathsf{E}(\mathscr{D}_\Gamma) := E, \qquad \mathsf{V}(\mathscr{D}_\Gamma) := Q, \qquad \mathsf{Ob}(\mathscr{D}_\Gamma) = \mathsf{E}(\mathscr{D}_\Gamma) \cup \mathsf{V}(\mathscr{D}_\Gamma).$$

To define the morphisms of $\mathscr{D}_\Gamma$ we define, for every $e \in E$, morphisms:

$$\mathsf{sor}(e) \xleftarrow{\ \mathsf{s}_e\ } e \xrightarrow{\ \mathsf{t}_e\ } \mathsf{tar}(e)$$

We complete the description of $\mathscr{D}_\Gamma$ by defining an identity morphism on each object of $\mathscr{D}_\Gamma$.

A morphism of D-categories induces a morphism of graphs and, conversely, a morphism of graphs induces a morphism of D-categories. Therefore, there are functors:

$$\begin{aligned} \mathsf{grph} : \mathsf{Dcat} &\to \mathsf{Grph} \\ \mathsf{dcat} : \mathsf{Grph} &\to \mathsf{Dcat}. \end{aligned}$$

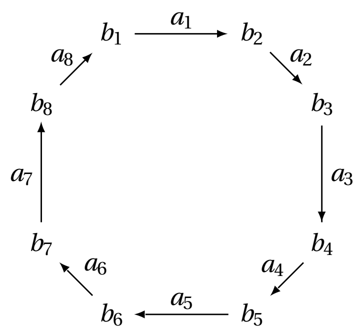
A more explicit formulation of this construction can be found in [2], where it was shown that:

Theorem 1.1. *There is an isomorphism of categories:*

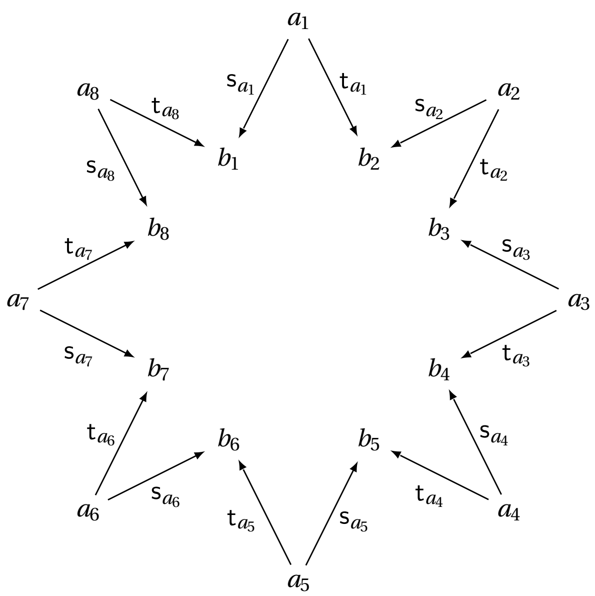
$$\mathsf{Dcat} \cong \mathsf{Grph},$$

where this isomorphism is given by the functor $\mathrm{grph} : \mathsf{Dcat} \to \mathsf{Grph}$ *with inverse* $\mathrm{dcat} : \mathsf{Grph} \to \mathsf{Dcat}$.

Example 1.1. For the graph given by:



The associated D-category is given by:



and vice-versa.

1.2 Hybrid Objects

With the notion of a D-category in hand, we can define hybrid objects over a category C.

Definition 1.3. Let C be a category. A *hybrid object over* C is a pair $(\mathscr{D}, \mathbf{A})$, where $\mathscr{D}$ is a D-category and

$$\mathbf{A} : \mathscr{D} \to \mathsf{C}$$

is a functor.

The functor $\mathbf{A}$ can be thought of as the *continuous component* of the hybrid object, and the category $\mathscr{D}$ its *discrete component*. The category $\mathsf{C}^{\mathscr{D}}$ is thus the category of hybrid objects over C with the same "discrete structure."

Example 1.2. Some specific examples of hybrid objects are given by:

Hybrid simplicial set	$\mathbf{K} : \mathscr{D} \to \mathsf{SSet}$
Hybrid topological space	$\mathbf{X} : \mathscr{D} \to \mathsf{Top}$
Hybrid chain complex	$\mathbf{A} : \mathscr{D} \to \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$
Hybrid manifold	$\mathbf{M} : \mathscr{D} \to \mathsf{Man}$

These will be investigated more in Chapter 3, where we will be interested in studying hybrid objects over a general model category M, i.e., functors $\mathbf{A} : \mathscr{D} \to \mathsf{M}$.

Let $\mathscr{A}$ and $\mathscr{B}$ be D-categories. For a morphism $\vec{F} : \mathscr{A} \to \mathscr{B}$ of D-categories, let

$$\vec{F}^* : \mathsf{C}^{\mathscr{B}} \to \mathsf{C}^{\mathscr{A}}$$

be the functor given on objects, i.e., functors $\mathbf{B} : \mathscr{B} \to \mathsf{C}$, and morphisms, i.e., natural transformations $\vec{g} : \mathbf{B} \to \mathbf{B}'$, of $\mathsf{C}^{\mathscr{B}}$ by $\vec{F}^*(\mathbf{B}) = \mathbf{B} \circ \vec{F}$ and $\vec{F}^*(\vec{g}) = \vec{g} \circ \vec{F}$.

Definition 1.4. Let $(\mathscr{A}, \mathbf{A})$ and $(\mathscr{B}, \mathbf{B})$ be two hybrid objects over the category C. A morphism of these hybrid objects, or just a *hybrid morphism*, is a pair

$$(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B}),$$

where $\vec{F} : \mathscr{A} \to \mathscr{B}$ is a morphism in Dcat and $\vec{f} : \mathbf{A} \stackrel{\bullet}{\to} \vec{F}^*(\mathbf{B})$ is a natural transformation in $\mathsf{C}^{\mathscr{A}}$.

1.2.1 Composing hybrid morphisms.

Given two hybrid morphisms $(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$ and $(\vec{G}, \vec{g}) : (\mathscr{B}, \mathbf{B}) \to (\mathscr{C}, \mathbf{C})$, the composite morphism is given by

$$(\vec{G}, \vec{g}) \circ (\vec{F}, \vec{f}) := (\vec{G} \circ \vec{F}, \vec{F}^*(\vec{g}) \bullet \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{C}, \mathbf{C}).$$

Specifically, the composite morphism is just the standard composition of functors and objectwise composition of natural transformations, i.e.,

$$\vec{F}^*(\vec{g}) \bullet \vec{f} : \mathbf{A} \stackrel{\bullet}{\to} (\vec{G} \circ \vec{F})^*(\mathbf{C}) = \vec{F}^*(\vec{G}^*(\mathbf{C})),$$

is defined objectwise by $(\vec{F}^*(\vec{g}) \bullet \vec{f})_a = \vec{F}^*(\vec{g})_a \circ \vec{f}_a = \vec{g}_{\vec{F}(a)} \circ \vec{f}_a$.

Definition 1.5. Let C be a category. The *category of hybrid objects over the category* C, denoted by $\mathsf{Hy}(\mathsf{C})$, has as

Objects: Hybrid objects over C, i.e., pairs $(\mathscr{A}, \mathbf{A})$, where $\mathbf{A} : \mathscr{A} \to \mathsf{C}$.
Morphisms: Morphisms of hybrid objects, i.e., pairs

$$(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B}),$$

where $\vec{F}$ is a morphism in Dcat and $\vec{f} : \mathbf{A} \stackrel{\cdot}{\to} \vec{F}^*(\mathbf{B})$ is a morphism in $\mathsf{C}^{\mathscr{A}}$.

For a functor $G : \mathsf{C} \to \mathsf{D}$ and a D-category $\mathscr{A}$, this induces a functor:

$$G_* : \mathsf{C}^{\mathscr{A}} \to \mathsf{D}^{\mathscr{A}}$$

given on functors $\mathbf{A} : \mathscr{A} \to \mathsf{C}$ by $G_*(\mathbf{A}) = G \circ \mathbf{A}$. For a natural transformations $\vec{f} : \mathbf{A} \stackrel{\cdot}{\to} \mathbf{A}'$ in $\mathsf{C}^{\mathscr{A}}$, $G_*(\vec{f})$ is defined objectwise by $G_*(\vec{f})_a := G(\vec{f}_a)$.

Using this, we can define:

1.2.2 Functors between categories of hybrid objects. A functor $G : \mathsf{C} \to \mathsf{D}$ between two categories induces a functor:

$$\mathsf{Hy}(G) : \mathsf{Hy}(\mathsf{C}) \to \mathsf{Hy}(\mathsf{D})$$

between categories of hybrid objects. On objects $(\mathscr{A}, \mathbf{A})$ and morphisms $(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$ of $\mathsf{Hy}(\mathsf{C})$, the functor $\mathsf{Hy}(G)$ is given by:

$$\begin{aligned} \mathsf{Hy}(G)(\mathscr{A}, \mathbf{A}) &= (\mathscr{A}, G_*(\mathbf{A})), \\ \mathsf{Hy}(G)(\vec{F}, \vec{f}) &= (\vec{F}, G_*(\vec{f})) : (\mathscr{A}, G_*(\mathbf{A})) \to (\mathscr{B}, G_*(\mathbf{B})). \end{aligned}$$

Functors between categories of hybrid objects will be investigated in detail in Chapter 2.

1.3 Hybrid Systems

The motivation for considering hybrid objects is given by hybrid systems. Systems of this form have been well-studied in the context of engineering, and have important applications to systems undergoing impacts, systems with non-smooth control laws and networks of dynamical systems. To better understand the relationship between hybrid systems and hybrid objects, we introduce the "standard" definition of a hybrid system (see [23] and the references therein).

Definition 1.6. A *hybrid system* is a tuple

$$\mathfrak{H} = (\Gamma, D, G, R, X),$$

where

- $\Gamma = (Q, E)$ is an oriented graph (possibly infinite).
- $D = \{D_i\}_{i\in Q}$ is a set of *domains* where D_i is a smooth manifold.
- $G = \{G_e\}_{e\in E}$ is a set of *guards*, where $G_e \subseteq D_{\mathsf{sor}(e)}$ is an embedded submanifold of $D_{\mathsf{sor}(e)}$.
- $R = \{R_e\}_{e\in E}$ is a set of *reset maps*; these are smooth maps $R_e : G_e \to D_{\mathsf{tar}(e)}$.
- $X = \{X_i\}_{i\in Q}$ is a collection of vector fields, i.e., X_i is a vector field on the manifold D_i.

1.3.1 Hybrid spaces. As with dynamical systems, it is sometimes desirable to consider the underlying "space" of a hybrid system. This amounts to "forgetting" about the vector field on each domain. More specifically, we can define a *smooth hybrid space* to be a tuple:

$$\mathbb{H} = (\Gamma, D, G, R).$$

Moreover, forgetting about the smooth structure on the elements of D, G and R, one can view $\mathbb{H}$ as a topological hybrid space. It will be seen that smooth hybrid spaces correspond to hybrid objects over the category of manifolds: hybrid manifolds. Hence, hybrid topological spaces correspond to hybrid objects over the category of topological spaces. We will later study hybrid topological spaces, through the notion of a hybrid object over a category. First, we motivate the study of hybrid systems by considering an example.

Example 1.3. The quintessential example of a hybrid system is given by the one-dimensional bouncing ball; see Figure 1.1. While this system has, arguably, been over-studied, we will utilize it in order to illustrate non-trivial ideas in a trivial setting.

A ball bouncing in one-dimension is naturally modeled as a hybrid system:

$$\mathfrak{H}^{\text{ball}} = (\Gamma^{\text{ball}}, D^{\text{ball}}, G^{\text{ball}}, R^{\text{ball}}, X^{\text{ball}}).$$

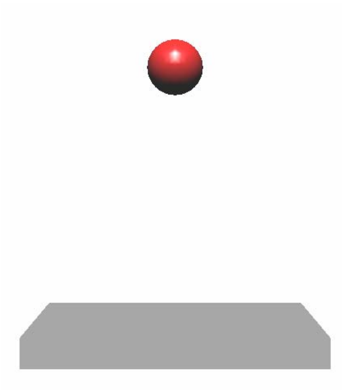
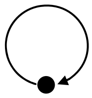


Figure 1.1: The bouncing ball.

That is, we consider a ball dropped from some positive height, say x_1, above a surface defined by $x_1 = 0$. Since the velocity of the ball will reset when it impacts the floor, the graph for this hybrid system is given by:

$$\Gamma^{\text{ball}} = (Q^{\text{ball}}, E^{\text{ball}}), \qquad Q^{\text{ball}} = \{1\}, \qquad E^{\text{ball}} = \{e = (1,1)\}.$$

That is, by a graph of the form:



Since the phase space of the bouncing ball will consist of two variables, the position x_1 and velocity x_2, the domain for the hybrid system is given by:

$$D_1^{\text{ball}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 \geq 0 \right\},$$

and $D^{\text{ball}} = \{D_1^{\text{ball}}\}$. The guard condition encodes the fact that a transition in the velocities of the system should occur when the position is zero and the velocity is "downward pointing." Therefore,

$$G_e^{\text{ball}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \leq 0 \right\},$$

and $G^{\text{ball}} = \{G_e^{\text{ball}}\}$. The reset map for the system is given by:

$$R_e^{\text{ball}}(x_1, x_2) = \begin{pmatrix} x_1 \\ -r x_2 \end{pmatrix},$$

where $0 \leq r \leq 1$ is the coefficient of restitution for the ball; this map encodes the fact that when the ball impacts the ground, its velocity is reversed and scaled down by the amount of energy lost through impact.

Finally, the vector field for this system is given by:

$$X_1^{\text{ball}}(x_1, x_2) = \begin{pmatrix} x_2 \\ -g \end{pmatrix},$$

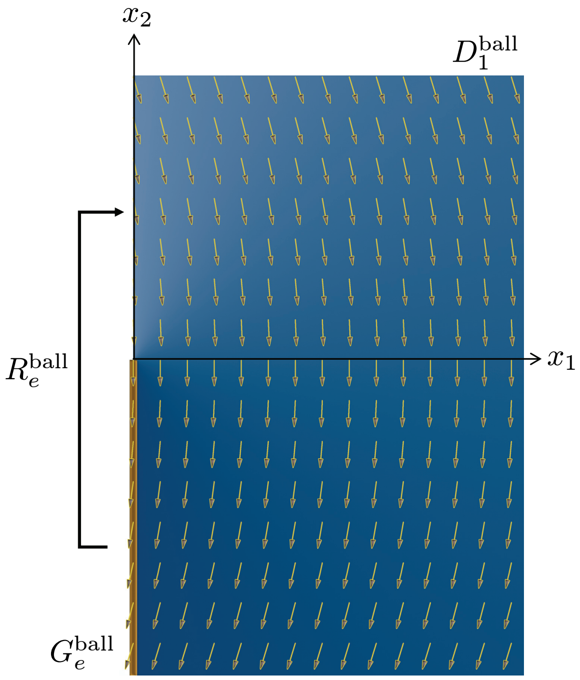


Figure 1.2: The hybrid model of a bouncing ball.

where g is the acceleration due to gravity. A graphical representation of this system can be seen in Figure 1.2.

1.3.2 Hybrid manifolds. We justify the notion of a hybrid object by relating hybrid manifolds to the hybrid space associated to a hybrid system.

A *hybrid manifold* is a hybrid object over the category of manifolds, i.e., a D-category $\mathscr{D}$ together with a functor:

$$\mathbf{M} : \mathscr{D} \to \mathsf{Man}. \tag{1.6}$$

In physical systems, it often is the case that for every $a \in \mathsf{E}(\mathscr{D})$, and hence every diagram

$$\begin{array}{ccc} & a & \\ {\scriptstyle \mathsf{s}_a} \swarrow & & \searrow {\scriptstyle \mathsf{t}_a} \\ \mathsf{cod}(\mathsf{s}_a) & & \mathsf{cod}(\mathsf{t}_a) \end{array}$$

in $\mathscr{D}$, the corresponding diagram in Man is given by:

$$\begin{array}{ccc} & \mathbf{M}_a & \\ {\scriptstyle \mathbf{M}_{\mathsf{s}_a} = \iota} \swarrow & & \searrow {\scriptstyle \mathbf{M}_{\mathsf{t}_a}} \\ \mathbf{M}_{\mathsf{cod}(\mathsf{s}_a)} & & \mathbf{M}_{\mathsf{cod}(\mathsf{t}_a)} \end{array} \tag{1.7}$$

where $\mathbf{M}_a \subseteq \mathbf{M}_{\mathsf{cod}(\mathsf{s}_a)}$ is an embedded submanifold and $\mathbf{M}_{\mathsf{s}_a} = \iota$ is the natural inclusion. We denote hybrid manifolds of this form by $\mathbf{M}^{\iota}$.

Although we do not explicitly assume that $\mathbf{M}_{\mathsf{s}_a}$ is an inclusion, this often is the case, as the following proposition indicates.

Proposition 1.1. *There is a bijective correspondence:*

$$\{\text{Hybrid Spaces, } \mathbb{H} = (\Gamma, D, G, R)\} \leftrightarrow \{\text{Hybrid Manifolds, } \mathbf{M}^{\iota} : \mathscr{D} \to \mathsf{Man})\}.$$

Proof. Given a hybrid space $\mathbb{H} = (\Gamma, D, G, R)$, we define the corresponding hybrid manifold by $\mathbf{M}^{(D,G,R)} : \mathscr{D}_\Gamma \to \mathsf{Man}$, where $\mathscr{D}_\Gamma = \mathrm{dcat}(\Gamma)$ is the D-category obtained from the graph Γ and $\mathbf{M}^{(D,G,R)}$ is defined for every $e \in \mathsf{E}(\mathscr{D}_\Gamma) = E$ by

$$\mathbf{M}^{(D,G,R)}\left(\begin{array}{ccc} & e & \\ {\scriptstyle \mathsf{s}_e}\swarrow & & \searrow{\scriptstyle \mathsf{t}_e} \\ \mathsf{sor}(e) & & \mathsf{tar}(e) \end{array} \right) := \begin{array}{ccc} & \mathbf{M}^{(D,G,R)}_e := G_e & \\ \mathbf{M}^{(D,G,R)}_{\mathsf{s}_e} := \iota \swarrow & & \searrow \mathbf{M}^{(D,G,R)}_{\mathsf{t}_e} := R_e \\ \mathbf{M}^{(D,G,R)}_{\mathsf{sor}(e)} := D_{\mathsf{sor}(e)} & & \mathbf{M}^{(D,G,R)}_{\mathsf{tar}(e)} := D_{\mathsf{tar}(e)} \end{array}$$

It is clear that $\mathbf{M}^{(D,G,R)}$ is a hybrid manifold.

Conversely, consider a hybrid manifold $\mathbf{M}^{\iota} : \mathscr{D} \to \mathsf{Man}$. Let $\Gamma_{\mathscr{D}} = \mathrm{grph}(\mathscr{D}) = (Q_{\mathscr{D}}, E_{\mathscr{D}})$ be the graph obtained from the D-category $\mathscr{D}$. We define

$$\mathbb{H}_{(\mathscr{D},\mathbf{M}^{\iota})} = (\Gamma_{\mathscr{D}}, D_{\mathbf{M}^{\iota}}, G_{\mathbf{M}^{\iota}}, R_{\mathbf{M}^{\iota}}),$$

where $D_{\mathbf{M}^{\iota}} := \{\mathbf{M}^{\iota}_b\}_{b \in \mathsf{V}(\mathscr{D}) = Q_{\mathscr{D}}}$, $G_{\mathbf{M}^{\iota}} := \{\mathbf{M}^{\iota}_a\}_{a \in \mathsf{E}(\mathscr{D}) = E_{\mathscr{D}}}$ and $R_{\mathbf{M}^{\iota}} := \{\mathbf{M}^{\iota}_{\mathsf{t}_a}\}_{a \in \mathsf{E}(\mathscr{D}) = E_{\mathscr{D}}}$. □

Example 1.4. The hybrid space for the bouncing ball is given by:

$$\mathbb{H}^{\text{ball}} = (\Gamma^{\text{ball}}, D^{\text{ball}}, G^{\text{ball}}, R^{\text{ball}}).$$

We will construct the associated hybrid object $(\mathscr{D}^{\text{ball}}, \mathbf{M}^{\text{ball}})$. The D-category associated with the graph Γ^{ball} is given by:

$$\mathscr{D}^{\text{ball}} = \begin{array}{c} a \\ {\scriptstyle \mathsf{s}_a}\downarrow\;\downarrow{\scriptstyle \mathsf{t}_a} \\ b \end{array}$$

together with the identity morphisms $\mathrm{id}_a : a \to a$ and $\mathrm{id}_b : b \to b$. The functor

$$\mathbf{M}^{\text{ball}} : \mathscr{D}^{\text{ball}} \to \mathsf{Man}$$

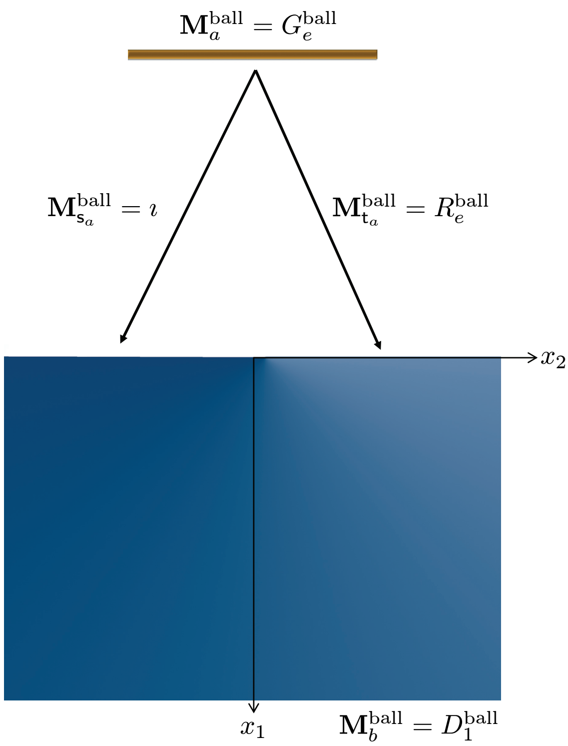


Figure 1.3: The hybrid manifold for the bouncing ball

takes the following values:

$$\mathbf{M}^{\text{ball}}\left(\begin{array}{c} a \\ \mathsf{s}_a \downarrow \downarrow \mathsf{t}_a \\ b \end{array}\right) = \begin{array}{c} \mathbf{M}_a^{\text{ball}} = G_e^{\text{ball}} \\ \mathbf{M}_{\mathsf{s}_a}^{\text{ball}} = \imath \downarrow \downarrow \mathbf{M}_{\mathsf{t}_a}^{\text{ball}} = R_e^{\text{ball}} \\ \mathbf{M}_b^{\text{ball}} = D_1^{\text{ball}} \end{array}$$

A graphical representation of this hybrid manifold can be seen in Figure 1.3

The original motivation for considering D-categories can now be seen; the edge sets of these categories serve the purpose of “pulling out the guard.” The claim is that small categories of any other shape would not allow for the representation of hybrid systems as functors in such a clear fashion.

Chapter 2

Adjunctions between Categories of Hybrid Objects

Central to category theory—and hence to all of mathematics—is the notion of a *universal property*, which characterizes objects that share a certain property, i.e., objects displaying such a property are unique up to isomorphism. Examples abound in category theory (e.g., products and limits) but also appear in engineering, although almost never recognized (e.g., stability is a universal property).

This chapter begins by reviewing some of the most fundamental universal constructions in a category: products, equalizers, pullbacks and limits. If all of these objects exist in a category, the category is said to be *complete*. We prove that Dcat is complete and $\mathsf{Hy}(\mathsf{C})$ is complete if C is complete.

The concept of a universal property is captured by *adjunctions* between categories. For example, the universality of the limit implies that it defines an adjunction:

$$\Delta_{\mathsf{J}} : \mathsf{C} \rightleftarrows \mathsf{C}^{\mathsf{J}} : \lim_{\mathsf{J}}.$$

After introducing the definition of an adjunction, we discuss the dual to limits, colimits, as defined by the adjunction:

$$\operatorname{colim}_{\mathsf{J}} : \mathsf{C}^{\mathsf{J}} \rightleftarrows \mathsf{C} : \Delta_{\mathsf{J}}.$$

If such an adjunction exists, C is said to be *cocomplete*. We demonstrate that Dcat is cocomplete and $\mathsf{Hy}(\mathsf{C})$ is cocomplete through the use of left Kan extensions. Finally, the adjunction:

$$\operatorname{hycolim} : \mathsf{Hy}(\mathsf{C}) \rightleftarrows \mathsf{Dcat} \times \mathsf{C} : \Delta^{\mathrm{hy}},$$

is introduced; this can be thought of as the hybrid analogue of the colimit adjunction.

After establishing the completeness and cocompleteness of categories of hybrid objects, which is the main result of this chapter, we turn our attention to adjunctions between categories of hybrid objects. Given an adjunction

$$L : \mathsf{C} \rightleftarrows \mathsf{D} : R$$

between categories, we demonstrate that there is a corresponding adjunction

$$\mathsf{Hy}(L) : \mathsf{Hy}(\mathsf{C}) \rightleftarrows \mathsf{Hy}(\mathsf{D}) : \mathsf{Hy}(R).$$

We conclude the chapter by relating categories of hybrid objects to fibered categories. This is achieved by considering the projection functor:

$$\Pi : \mathsf{Hy}(\mathsf{C}) \to \mathsf{Dcat},$$

which defines a *fibration*. In particular, a fiber of $\mathsf{Hy}(\mathsf{C})$ is given by $\mathsf{Hy}(\mathsf{C})_{\mathscr{D}} \cong \mathsf{C}^{\mathscr{D}}$. This allows us to conclude that a pair of *cartesian* functors

$$L : \mathsf{Hy}(\mathsf{C}) \rightleftarrows \mathsf{Hy}(\mathsf{D}) : R$$

are adjoint if they are adjoint *fiberwise*, i.e., on each fiber.

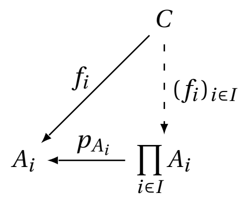
2.1 Limits in Categories of Hybrid Objects

The main objective of this section is to prove that both Dcat and Hy(C) are complete, i.e., limits exist, if C is complete. In fact, the limit of a diagram of hybrid objects is obtained through the limit in Dcat and the limit in C. We first review the more elementary notions of products, equalizers and pullbacks in order to motivate the subsequent constructions.

2.1.1 Products. Let C be a category, I be a set and $\{A_i\}_{i\in I}$ a set of objects of C. The product of these objects is an object $\prod_{i\in I} A_i$ of C together with *projections* $p_{A_i} : \prod_{i\in I} A_i \to A_i$ satisfying the universal property that for any other object C of C with morphisms $f_i : C \to A_i$, there exists a unique morphism

$$(f_i)_{i\in I} : C \to \prod_{i\in I} A_i$$

making the following diagram

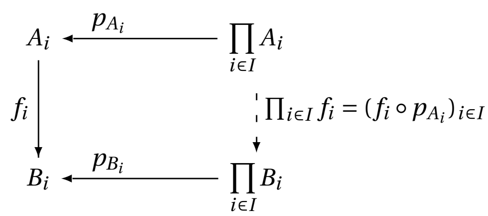


commute for all $i \in I$.

Given two sets of objects $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ together with morphisms $f_i : A_i \to B_i$, there is a unique induced morphism

$$\prod_{i\in I} f_i = (f_i \circ p_{A_i})_{i\in I} : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$$

making the following diagram



commute.

Definition 2.1. A category C is said to *have products*, or products *exist* in C, if for any set of objects $\{A_i\}_{i\in I}$ in C, the product $\prod_{i\in I} A_i$ exists.

Remark 2.1. Products of the form given in Definition 2.1 are often referred to as *small products*. Sometimes finite products—I is a finite set—are often of interest.

Example 2.1. The category of sets, Set, has products. For a set of sets $\{X_i\}_{i\in I}$, the product is the usual cartesian product:

$$\prod_{i\in I} X_i = \{(x_i)_{i\in I} : x_i \in X_i\}.$$

The projections are defined as

$$\begin{array}{rcl} p_i : \prod_{i\in I} X_i & \to & X_i \\ (x_i)_{i\in I} & \mapsto & x_i. \end{array}$$

To verify the universal property of the product, consider a set D and functions $f_i : D \to X_i$. From these we obtain a function $f : D \to \prod_{i\in I} X_i$ given by:

$$f(y) = (f_i(y))_{i\in I}$$

for all $y \in D$.

2.1.2 Products of graphs. The category of graphs, Grph, has products. For a set of graphs $\{\Gamma_i = (Q_i, E_i)\}_{i\in I}$, the product is induced from the product on sets as follows:

$$\prod_{i\in I} \Gamma_i = (\prod_{i\in I} Q_i, \prod_{i\in I} E_i).$$

The source and target maps for the product graph $\prod_{i\in I} \Gamma_i$ are defined to be the unique maps making the following diagrams commute:

$$\begin{array}{ccc} E_i & \xleftarrow{p_{E_i}} & \prod_{i\in I} E_i \\ \downarrow{\scriptstyle \mathsf{sor}_i} & & \downarrow{\scriptstyle \prod_{i\in I}\mathsf{sor}_i} \\ Q_i & \xleftarrow{p_{Q_i}} & \prod_{i\in I} Q_i \end{array} \qquad \begin{array}{ccc} E_i & \xleftarrow{p_{E_i}} & \prod_{i\in I} E_i \\ \downarrow{\scriptstyle \mathsf{tar}_i} & & \downarrow{\scriptstyle \prod_{i\in I}\mathsf{tar}_i} \\ Q_i & \xleftarrow{p_{Q_i}} & \prod_{i\in I} Q_i \end{array}$$

Specifically, for $(e_i)_{i\in I} \in \prod_{i\in I} E_i$,

$$\prod_{i\in I} \mathsf{sor}_i((e_i)_{i\in I}) = (\mathsf{sor}_i(e_i))_{i\in I}, \qquad \prod_{i\in I} \mathsf{tar}_i((e_i)_{i\in I}) = (\mathsf{tar}_i(e_i))_{i\in I}.$$

Note that the projections are defined by:

$$p_{\Gamma_i} := (p_{E_i}, p_{Q_i}) : \prod_{i\in I} \Gamma_i \to \Gamma_i.$$

Finally, we must verify the universal property of the product. Consider a graph $\Gamma = (Q, E)$ together with a collection of morphisms $F_i = ((F_Q)_i, (F_E)_i) : \Gamma \to \Gamma_i$. It follows from the universality of products in the category of sets that the diagrams given in Table 2.1 are commutative. So $F = (F_Q, F_E) : \Gamma \to \prod_{i\in I} \Gamma_i$ is the desired unique morphism.

2.1.3 Products in categories of hybrid objects. The existence of products in C relates to the existence of products in $\mathsf{Hy}(\mathsf{C})$. In order to establish this relationship, we need to show that products exist in Dcat and that if products exist for C then they exist for C^{J} for any small category J. These two results are then "glued" together to yield products in $\mathsf{Hy}(\mathsf{C})$.

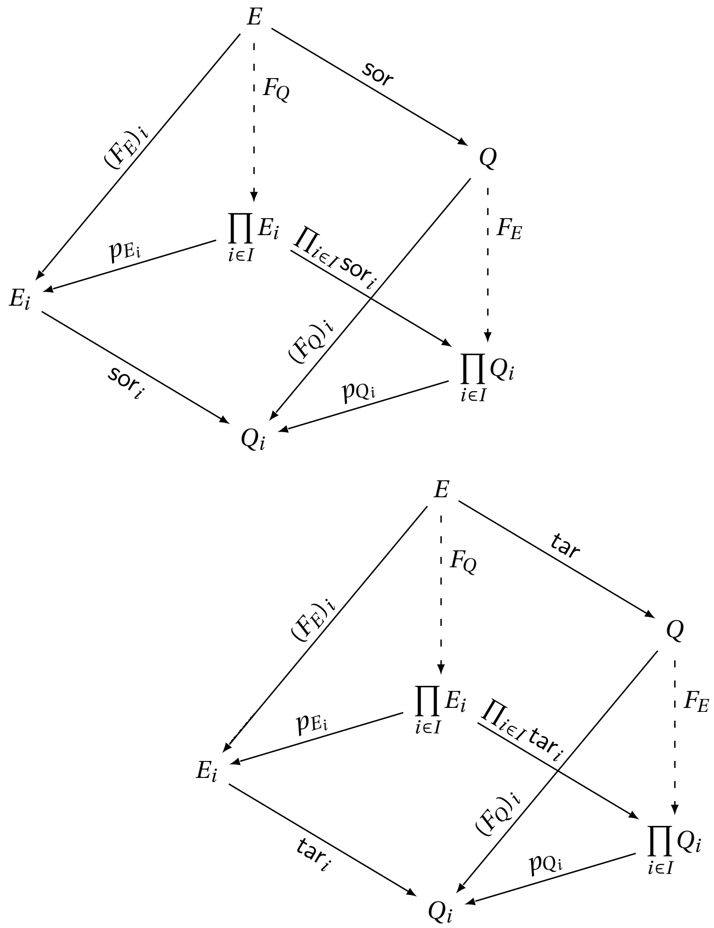


Table 2.1: Commuting diagrams verifying the universality of the product in Grph.

Proposition 2.1. *Products exist in* Dcat*. Specifically, for* $\{\mathscr{A}_i\}_{i\in I}$ *a set of D-categories, the product* $\prod_{i\in I}\mathscr{A}_i$ *exists and is given by:*

$$\prod_{i\in I}\mathscr{A}_i = \mathrm{dcat}(\prod_{i\in I}\mathrm{grph}(\mathscr{A}_i)),$$

where $\prod_{i\in I}\mathrm{grph}(\mathscr{A}_i)$ *is the product of graphs.*

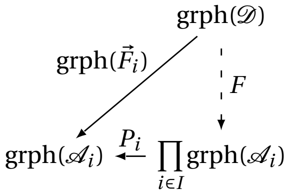
Proof. This follows from the fact that dcat and grph are isomorphisms between categories (Theorem 1.1). Specifically, the projections:

$$P_i : \prod_{i\in I}\mathrm{grph}(\mathscr{A}_i) \to \mathrm{grph}(\mathscr{A}_i)$$

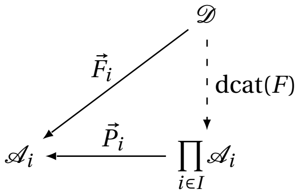
yield projections of D-categories:

$$\vec{P}_i := \mathrm{dcat}(P_i) : \prod_{i\in I}\mathscr{A}_i = \mathrm{dcat}(\prod_{i\in I}\mathrm{grph}(\mathscr{A}_i)) \to \mathscr{A}_i = \mathrm{dcat}(\mathrm{grph}(\mathscr{A}_i)).$$

Now, to verify universality, for any other D-category $\mathscr{D}$ with morphisms $\vec{F}_i : \mathscr{D} \to \mathscr{A}_i$ there is a graph $\mathrm{grph}(\mathscr{D})$ and morphisms $\mathrm{grph}(\vec{F}_i) : \mathrm{grph}(\mathscr{D}) \to \mathrm{grph}(\mathscr{A}_i)$. By the universality of the product in Grph, there exists a unique morphism F making the following diagram



commute. Applying the functor dcat yields a commuting diagram



where dcat(F) must be unique; if there were another morphisms making the diagram commute, it would also make the corresponding diagram of graphs commute, thus violating the uniqueness of F. □

Lemma 2.1. *If products exist in* C, *then products exists in* C^{J} *for any small category* J. *Specifically, for a set of functors* $F_i : \mathsf{J} \to \mathsf{C}$, $i \in I$, *the product is given on objects and morphisms by:*

$$(\prod_{i\in I} F_i)(a) = \prod_{i\in I} F_i(a), \qquad (\prod_{i\in I} F_i)(\alpha) = \prod_{i\in I} F_i(\alpha).$$

Proof. See [18], Theorem 1, page 115. □

Proposition 2.2. *If products exist in* C, *then products exist in* $\mathsf{Hy}(\mathsf{C})$. *Specifically, for a set of hybrid objects* $\{(\mathscr{A}_i, \mathbf{A}_i)\}_{i\in I}$, *the product is given by:*

$$\prod_{i\in I}(\mathscr{A}_i, \mathbf{A}_i) = (\prod_{i\in I}\mathscr{A}_i, \prod_{i\in I}\vec{P}_i^*(\mathbf{A}_i))$$

where $\prod_{i\in I}\mathscr{A}_i$ *is the product of D-categories,* $\prod_{i\in I}\vec{P}_i^*(\mathbf{A}_i)$ *is the product in* $\mathsf{C}^{\prod_{i\in I}\mathscr{A}_i}$ *with* $\vec{P}_i : \prod_{i\in I}\mathscr{A}_i \to \mathscr{A}_i$ *the projection morphisms in* Dcat.

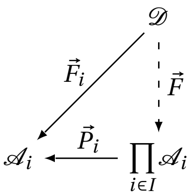
Proof. The projection morphisms are given by:

$$(\vec{P}_i, \vec{p}_i) : \prod_{i\in I}(\mathscr{A}_i, \mathbf{A}_i) \to (\mathscr{A}_i, \mathbf{A}_i),$$

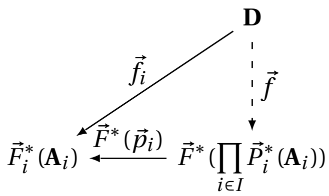
where $\vec{P}_i : \prod_{i\in I}\mathscr{A}_i \to \mathscr{A}_i$ and

$$\vec{p}_i : \prod_{i\in I}\vec{P}_i^*(\mathbf{A}_i) \dot{\to} \vec{P}_i^*(\mathbf{A}_i)$$

is objectwise the projection in C. We must verify the universality of the product. Consider a hybrid object $(\mathscr{D}, \mathbf{D})$ together with morphisms $(\vec{F}_i, \vec{f}_i) : (\mathscr{D}, \mathbf{D}) \to (\mathscr{A}_i, \mathbf{A}_i)$. By the universality of the product in Dcat, there exists a unique morphism $\vec{F} : \mathscr{D} \to \prod_{i\in I}\mathscr{A}_i$ yielding a commuting diagram



Therefore, we need only find a unique natural transformation $\vec{f} : \mathbf{D} \dot{\to} \vec{F}^*(\prod_{i\in I} \vec{P}_i^*(\mathbf{A}_i))$ in $\mathsf{C}^{\mathscr{D}}$. Since $\vec{f}_i : \mathbf{D} \dot{\to} \vec{F}_i^*(\mathbf{A}_i)$ there is a commuting diagram



in $\mathsf{C}^{\mathscr{D}}$ where the existence and uniqueness of $\vec{f}$ follows from the universal property of the product in $\mathsf{C}^{\mathscr{D}}$. □

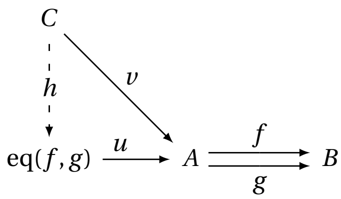
2.1.4 Equalizers. For a category C and a pair of morphisms:

$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

between two object A and B of C, the equalizer of this pair is an object $\mathrm{eq}(f,g)$ of C together with a morphism $u : \mathrm{eq}(f,g) \to B$ making diagram

$$\mathrm{eq}(f,g) \xrightarrow{u} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

commute, i.e., $f \circ u = g \circ u$. In addition, it must satisfying the universal property that for any other object C with a morphism $v : C \to A$ such that $f \circ v = g \circ v$, there exists a unique morphism $h : C \to \mathrm{eq}(f,g)$ such that the following diagram commutes:



Definition 2.2. A category C is said to *have equalizers* if for any pair of morphism $f, g : A \to B$ between any pair of objects in C, the equalizer exists.

Example 2.2. In the category of sets, Set, equalizers exist. For two sets X and Y and two functions $f, g : X \to Y$, the equalizer is given by:

$$\mathrm{eq}(f,g) = \{x \in X : f(x) = g(x)\},$$

with $u : \mathrm{eq}(f,g) \to X$ the inclusion. For a set Z and a morphism $h : Z \to X$ such that $f \circ h = g \circ h$, then $h : Z \to \mathrm{eq}(f,g)$ by the definition of $\mathrm{eq}(f,g)$, and hence is unique.

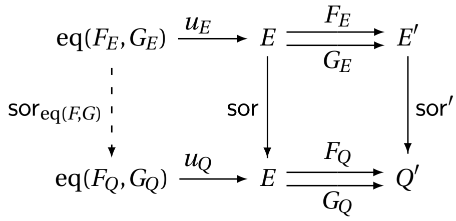
2.1.5 Equalizers in Grph**.** For a diagram in Grph of the form:

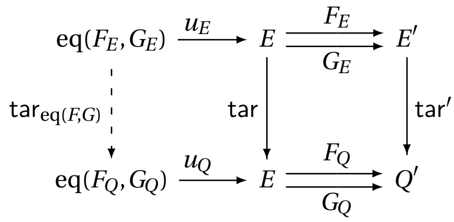
$$\Gamma = (Q,E) \overset{F=(F_Q,F_E)}{\underset{G=(G_Q,G_E)}{\rightrightarrows}} \Gamma' = (Q',E'),$$

the equalizer of this pair of morphisms exists. It is given by:

$$\mathrm{eq}(F,Q) = (\mathrm{eq}(F_Q, G_Q), \mathrm{eq}(F_E, G_E)),$$

where the equalizers on the right are in the category of sets. The source and target functions for $\mathrm{eq}(F,G)$ are given uniquely by requiring that the following diagrams commute:





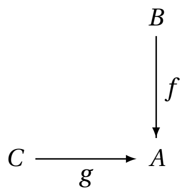
Note that the uniqueness of the source and target functions are due to the universality of equalizers in Set. It also follows from the definition of equalizers in Set that

$$\mathsf{sor}_{\mathrm{eq}(F,G)} = \mathsf{sor}|_{\mathrm{eq}(F_E,G_E)}, \qquad \mathsf{tar}_{\mathrm{eq}(F,G)} = \mathsf{tar}|_{\mathrm{eq}(F_E,G_E)},$$

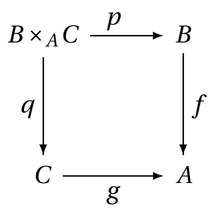
since u_E and u_Q are inclusions.

The universality of the equalizer in Grph is easy to verify (it is a simple exercise in diagram chasing).

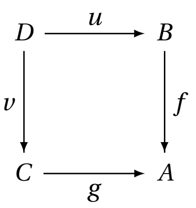
2.1.6 Pullbacks. Consider a category C and a diagram of the form:



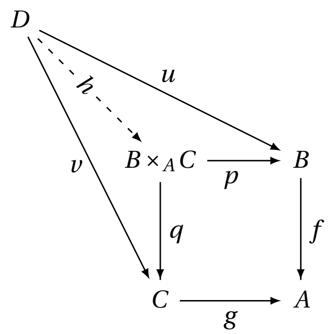
The pullback of this diagram is an object $B \times_A C$ of C together with two morphisms p and q such that the following *pullback* diagram



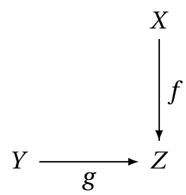
commutes. It is universal in the following sense: for any other object D of C with morphisms u and v making the following diagram commute



there exists a unique morphism $h : D \to B \times_A C$ such that the following diagram commutes:



Example 2.3. In the category of sets, pullbacks exists. Specifically, for a diagram of sets of the form:



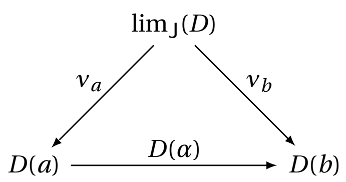
The pullback is given by:

$$X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

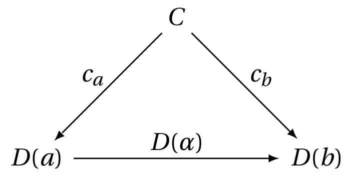
2.1.7 Limits. For a category C and a functor $D : \mathsf{J} \to \mathsf{C}$ the limit, if it exists, is an object of C, denoted by $\lim_{\mathsf{J}}(D)$, together with morphisms:

$$\nu_a : \lim_{\mathsf{J}}(D) \to D(a), \qquad a \in \mathsf{Ob}(\mathsf{J}),$$

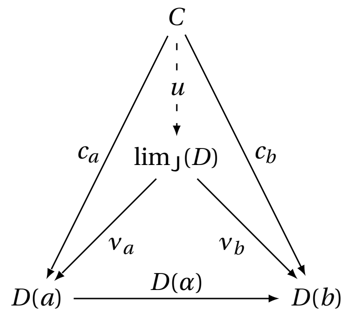
such that for every $\alpha : a \to b$ in J, the following diagram



commutes. In addition it is required to satisfy the universal property that for any object C of C with morphisms $c_a : C \to D(a)$, $a \in \mathsf{Ob}(\mathsf{J})$, such that there is a commuting diagram:



there exists a unique morphism $u : C \to \lim_{\mathsf{J}}(D)$ making the following diagram

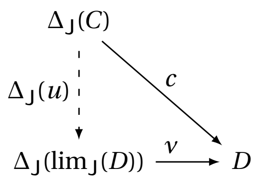


commute.

The notion of a limit perhaps can be better understood utilizing the language of natural transformations. For the constant functor $\Delta_{\mathsf{J}} : \mathsf{C} \to \mathsf{C}^{\mathsf{J}}$, the limit of D is an object $\lim_{\mathsf{J}}(D)$ of C together with a *universal* natural transformation:

$$\nu : \Delta_{\mathsf{J}}(\lim_{\mathsf{J}}(D)) \xrightarrow{\cdot} D.$$

It must be *universal* in the following sense: for any other object C of C and natural transformation $c : \Delta_{\mathsf{J}}(C) \xrightarrow{\cdot} D$, there exists a unique morphism $u : C \to \lim_{\mathsf{J}}(D)$ such that the following diagram commutes



in C^{J}.

Definition 2.3. A category C is *complete* if for every small category J and every functor $D : \mathsf{J} \to \mathsf{C}$, the limit exists.

There is the following useful lemma:

Lemma 2.2. *If C is complete, then it has a terminal object.*

Example 2.4. The category of sets, Set, is the canonical example of a complete category.

The category of small categories Cat is complete; this completeness is directly a result of the completeness of Set. In fact, one might be tempted to say that the category of D-categories is complete since the category of small categories is complete. The problem with this logic is that there is no guarantee that the limit of diagram in Dcat is again in Dcat.

Another example of a complete category is the category of graphs, Grph, which is again complete because the category of sets is complete. It turns out that the completeness of this category does imply the completeness of Dcat, which is not surprising in light of the isomorphism Dcat $\cong$ Grph.

2.1.8 Special cases of the limit. The limit includes as a special case all of the previous universal constructions we have introduced. Specifically,

Products. The limit of a functor

$$D : \mathsf{I} \to \mathsf{C},$$

where I is the discrete category obtained from an indexing set I.

Equalizers. The limit of a functor

$$D : (\bullet \rightrightarrows \bullet) \to \mathsf{C}.$$

Pullback. The limit of a functor

$$D : (\bullet \to \bullet \leftarrow \bullet) \to \mathsf{C}.$$

Interestingly enough, the existence of limits in a category is related to the existence of equalizers and products.

Proposition 2.3. *A category* C *is complete iff it has equalizers and products.*

Proof. See Corollary 2, page 113, [18]. □

Corollary 2.1. *The category of graphs,* Grph*, is complete.*

A corollary of this is that the category of D-categories is complete. Before stating this result, we introduce some notation.

Notation 2.1. To differentiate, when necessary, between limits in different categories, we sometimes write $\lim_{\mathsf{J}}^{\mathsf{C}}$ for the limit of a functor $D : \mathsf{J} \to \mathsf{C}$. Similarly, we sometimes write $\Delta_{\mathsf{J}}^{\mathsf{C}}$.

Theorem 2.1. *The category of D-categories,* Dcat*, is complete. Specifically, for a functor* $D : \mathsf{J} \to \mathsf{Dcat}$*,* J *small, the limit is given by:*

$$\lim_{\mathsf{J}}^{\mathsf{Dcat}}(D) = \mathrm{dcat}\left(\lim_{\mathsf{J}}^{\mathsf{Grph}}(\mathrm{grph}_*(D))\right).$$

Proof. Follows from the fact that dcat and grph are isomorphisms of categories; the proof is analogous to the proof of Proposition 2.1. □

2.1.9 The limit as a functor. If C is a complete category, then the limit exists for every diagram over a small category J, i.e., for every functor $D : \mathsf{J} \to \mathsf{C}$. In fact, the universality of the limit implies that it defines a functor

$$\lim_{\mathsf{J}} : \mathsf{C}^{\mathsf{J}} \to \mathsf{C}.$$

Specifically, consider two functors $D, D' : \mathsf{J} \to \mathsf{C}$ together with the corresponding universal natural transformations:

$$\begin{aligned} \nu : \Delta_{\mathsf{J}}(\lim_{\mathsf{J}}(D)) &\;\dot{\to}\; D, \\ \nu' : \Delta_{\mathsf{J}}(\lim_{\mathsf{J}}(D')) &\;\dot{\to}\; D'. \end{aligned}$$

The object function of the limit (as a functor) associates to these functors their limit. For a morphism $f : D \dot{\to} D'$, the limit of this morphism is the unique morphism $\lim_{\mathsf{J}}(f) : \lim_{\mathsf{J}}(D) \to \lim_{\mathsf{J}}(D')$ making the following diagram:

$$\begin{array}{ccc} \Delta_{\mathsf{J}}(\lim_{\mathsf{J}}(D)) & \xrightarrow{\nu} & D \\ {\scriptstyle \Delta_{\mathsf{J}}(\lim_{\mathsf{J}}(f))}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \Delta_{\mathsf{J}}(\lim_{\mathsf{J}}(D')) & \xrightarrow{\nu'} & D' \end{array}$$

commute.

Proposition 2.4. *If C is complete, then C^{K} is complete for every small category K. Specifically, for $D : \mathsf{J} \to \mathsf{C}^{\mathsf{K}}$, the limit is given on objects and morphisms of K by:*

$$\lim_{\mathsf{J}}^{\mathsf{C}^{\mathsf{K}}}(D)(a) = \lim_{\mathsf{K}}^{\mathsf{C}}(D(a)), \qquad \lim_{\mathsf{J}}^{\mathsf{C}^{\mathsf{K}}}(D)(\alpha) = \lim_{\mathsf{K}}^{\mathsf{C}}(D(\alpha)).$$

Proof. See [18], Theorem 1, page 115. □

2.1.10 Diagrams in categories of hybrid objects. By slight abuse of notation, we denote a diagram in $\mathsf{Hy}(\mathsf{C})$ by

$$(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{Hy}(\mathsf{C}).$$

That is, for every $\alpha : a \to b$ in J, there are corresponding hybrid objects and morphisms:

$$(\mathscr{D}^{\mathsf{J}}(a), \mathbf{D}^{\mathsf{J}}(a)) \xrightarrow{(\mathscr{D}^{\mathsf{J}}(\alpha), \mathbf{D}^{\mathsf{J}}(\alpha))} (\mathscr{D}^{\mathsf{J}}(b), \mathbf{D}^{\mathsf{J}}(b)).$$

In particular, $\mathscr{D}^{\mathsf{J}}(\alpha) : \mathscr{D}^{\mathsf{J}}(a) \to \mathscr{D}^{\mathsf{J}}(b)$ is a morphism of D-categories and

$$\mathbf{D}^{\mathsf{J}}(\alpha) : \mathbf{D}^{\mathsf{J}}(a) \dot{\to} (\mathscr{D}^{\mathsf{J}}(\alpha))^{*}(\mathbf{D}^{\mathsf{J}}(b))$$

is a morphism in $\mathsf{C}^{\mathscr{D}^{\mathsf{J}}(a)}$.

Note that by the definition of a hybrid object, we can without ambiguity write $\mathscr{D}^{\mathsf{J}} : \mathsf{J} \to \mathsf{Dcat}$; note that $\mathscr{D}^{\mathsf{J}}$ is *not* a D-category, but a diagram of such categories. Since the category of D-categories is complete, there exists a D-category $\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})$ together with a universal natural transformation:

$$\vec{V} : \Delta_{\mathsf{J}}^{\mathsf{Dcat}}(\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})) \dot{\to} \mathscr{D}^{\mathsf{J}}.$$

The motivation for denoting this natural transformation by $\vec{V}$ is that for every diagram of the form $\alpha : a \to b$ in J, there is a diagram of D-categories:

$$\begin{array}{ccc} & \lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}}) & \\ \swarrow^{\vec{V}_a} & & \searrow^{\vec{V}_b} \\ \mathscr{D}^{\mathsf{J}}(a) & \xrightarrow{\mathscr{D}^{\mathsf{J}}(\alpha)} & \mathscr{D}^{\mathsf{J}}(b) \end{array} \tag{2.1}$$

For the diagram $(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{Hy}(\mathsf{C})$ in $\mathsf{Hy}(\mathsf{C})$, the limit of $\mathscr{D}^{\mathsf{J}} : \mathsf{J} \to \mathsf{Dcat}$ yields a functor:

$$\vec{V}^*(\mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{C}^{\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}$$

defined on objects and morphisms of J by:

$$\vec{V}^*(\mathbf{D}^{\mathsf{J}})(a) := \vec{V}_a^*(\mathbf{D}^{\mathsf{J}}(a)), \qquad \vec{V}^*(\mathbf{D}^{\mathsf{J}})(\alpha) := \vec{V}^*_{\mathsf{dom}(\alpha)}(\mathbf{D}^{\mathsf{J}}(\alpha)).$$

Note that $\vec{V}^*(\mathbf{D}^{\mathsf{J}})$ is well-defined because of the commutativity of (2.1).

Using this notation, we can now prove that categories of hybrid objects are complete and give an explicit formula for the limit of a diagram.

Theorem 2.2. *If* C *is complete, then* $\mathsf{Hy}(\mathsf{C})$ *is complete. Specifically, for* $(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{Hy}(\mathsf{C})$*, the limit is given by:*

$$\lim_{\mathsf{J}}^{\mathsf{Hy}(\mathsf{C})}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) = \left(\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}}), \lim_{\mathsf{J}}^{\mathsf{C}^{\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{V}^*(\mathbf{D}^{\mathsf{J}}))\right).$$

Proof. The first step is to find the universal natural transformation in $\mathsf{Hy}(\mathsf{C})^{\mathsf{J}}$

$$\nu : \Delta_{\mathsf{J}}^{\mathsf{Hy}(\mathsf{C})}\left(\lim_{\mathsf{J}}^{\mathsf{Hy}(\mathsf{C})}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}})\right) \stackrel{\cdot}{\to} (\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}).$$

There are universal natural transformations

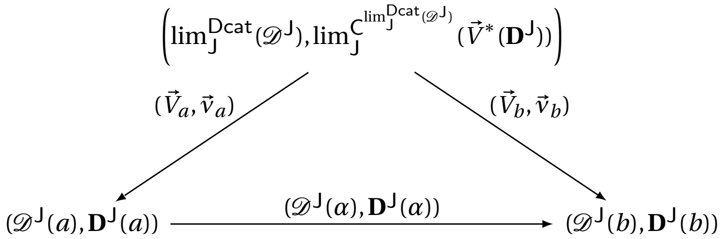
$$\begin{array}{rcl} \vec{V} : \Delta_{\mathsf{J}}^{\mathsf{Dcat}}\left(\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})\right) & \stackrel{\cdot}{\to} & \mathscr{D}^{\mathsf{J}} \\ \vec{\nu} : \Delta_{\mathsf{J}}^{\mathsf{C}^{\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}\left(\lim_{\mathsf{J}}^{\mathsf{C}^{\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{V}^*(\mathbf{D}^{\mathsf{J}}))\right) & \stackrel{\cdot}{\to} & \vec{V}^*(\mathbf{D}^{\mathsf{J}}) \end{array}$$

in $\mathsf{Dcat}^{\mathsf{J}}$ and $(\mathsf{C}^{\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})})^{\mathsf{J}}$.

The claim is that the universal transformation ν is given by $\nu = (\vec{V}, \vec{\nu})$. To verify this, first note that for an object a of J,

$$\begin{aligned} \nu_a = (\vec{V}_a, \vec{\nu}_a) : \lim_{\mathsf{J}}^{\mathsf{Hy}(\mathsf{C})}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) = \left(\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}}), \lim_{\mathsf{J}}^{\mathsf{C}^{\lim_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{V}^*(\mathbf{D}^{\mathsf{J}}))\right) \\ \stackrel{\cdot}{\to} (\mathscr{D}^{\mathsf{J}}(a), \mathbf{D}^{\mathsf{J}}(a)) \end{aligned}$$

since $\vec{V}_a^*(\mathbf{D}^{\mathsf{J}}(a)) = \vec{V}^*(\mathbf{D}^{\mathsf{J}})(a)$. Now, we need to verify that defining $\nu = (\vec{V}, \vec{\nu})$ in fact yields a natural transformation. That is, for $\alpha : a \to b$ in J, we need to show that there is a commuting diagram:

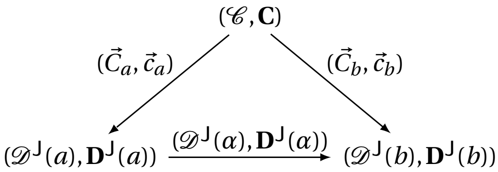


By the commutativity of (2.1), this follows from the fact that

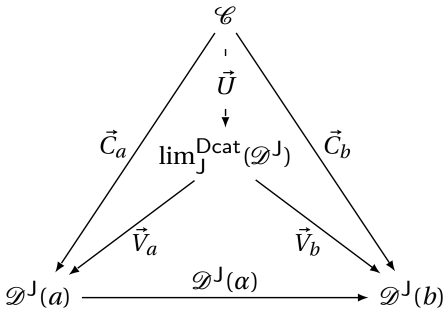
$$\vec{V}_a^*(\mathbf{D}^{\mathsf{J}}(\alpha)) \bullet \vec{v}_a = \vec{V}^*(\mathbf{D}^{\mathsf{J}})(\alpha) \bullet \vec{v}_a = \vec{v}_b,$$

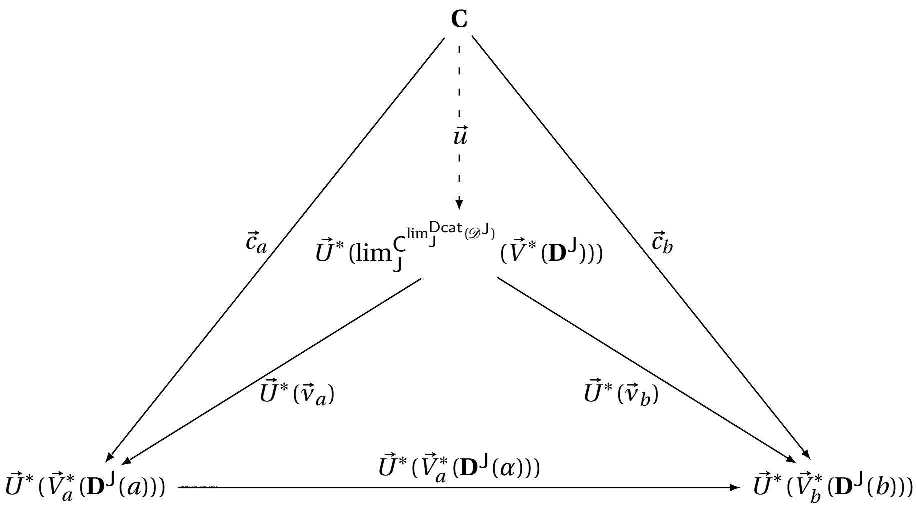
which is implied by the naturality of $\vec{v}$ and the definition of $\vec{V}^*(\mathbf{D}^{\mathsf{J}})$.

To conclude, we need only show the universality of $v = (\vec{V}, \vec{v})$. Suppose that there is a hybrid object $(\mathscr{C}, \mathbf{C})$ together with a collection of morphisms $(\vec{C}_a, \vec{c}_a) : (\mathscr{C}, \mathbf{C}) \to (\mathscr{D}^{\mathsf{J}}(a), \mathbf{D}^{\mathsf{J}}(a))$ of hybrid objects making the following following diagram



commute. This yields commuting diagrams and unique morphisms:

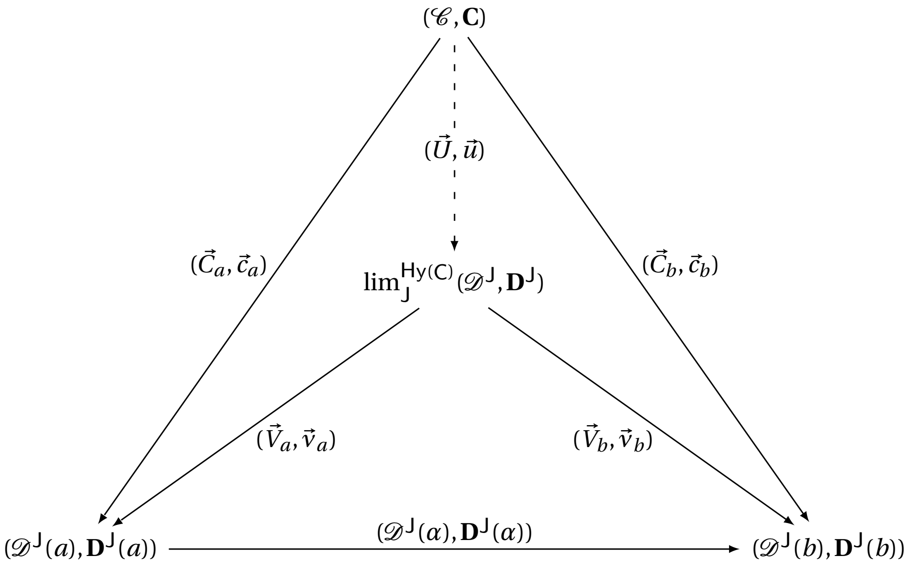




That is, we obtain a unique morphism of hybrid objects:

$$(\vec{U}, \vec{u}) : (\mathscr{C}, \mathbf{C}) \to \lim_{\mathsf{J}}^{\mathsf{Hy(C)}} (\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}})$$

that makes the following diagram



commute as desired. □

2.2 Adjunctions

Adjunctions between categories capture the notion of universality, and are one of the most fundamental constructions in category theory. All of the previous universal objects:

Products, Equalizers, Pullbacks, Limits,

can be formulated using the notion of an adjunction. In addition, adjunctions allow one to simplify many proofs in category theory. This will be seen in the next section where proving that Dcat is cocomplete through this use of adjunctions is a trivial matter.

Definition 2.4. Let C and D be two categories. A pair of adjoint functors $L\colon \mathsf{C} \to \mathsf{D}$ and $R\colon \mathsf{D} \to \mathsf{C}$, denoted by:

$$L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R$$

are functors such that there exists a natural bijection:

$$\varphi_{X,A}\colon \mathsf{Hom}_{\mathsf{D}}(L(X), A) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{C}}(X, R(A)).$$

for every object $X \in \mathsf{Ob}(\mathsf{C})$ and $A \in \mathsf{Ob}(\mathsf{D})$.

The functor L is termed the *left adjoint to* R and, equivalently, the functor R is termed the *right adjoint* to L.

Notation 2.2. There are variations of notation related to adjoint functors. The most important differences are related to the "directionality" of adjunctions; this is encoded in the fact that there are left adjoints and right adjoints. For example, in [18], an adjunction is a tuple $\langle L, R, \varphi\rangle$ with the elements of this tuple defined as in 2.4. The directionality of this adjunction, i.e., the fact that L is left adjoint to R (and so R is right adjoint to L) is stressed by denoting an adjunction by:

$$\langle L, R, \varphi\rangle : \mathsf{C} \rightharpoonup \mathsf{D}. \tag{2.2}$$

This also serves the dual purpose of making explicit the source and target categories of the functors L and R.

All this being said, we will denote an adjunction (or adjoint pair), as in Definition 2.4, simply by:

$$L\colon \mathsf{C} \rightleftarrows \mathsf{D}\colon R \tag{2.3}$$

From this notation, we infer that L is left adjoint to R (and so R is right adjoint to L). Moreover, implicit in this notation is the existence of a natural bijection φ which, when necessary, may be explicitly stated. In the case when it is too notation intensive to introduce the natural bijections associated to an adjunction, we will write:

$$f^\sharp = \varphi_{X,A}(f) : X \to R(A), \qquad g^\flat = \varphi^{-1}_{X,A}(g) : L(X) \to A,$$

for $f : L(X) \to A$ and $g : X \to R(A)$.

2.2.1 Natural bijections. It is important to understand what it means for

$$\varphi_{X,A} : \mathsf{Hom}_{\mathsf{D}}(L(X), A) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{C}}(X, R(A)) \tag{2.4}$$

to be a *natural bijection.* First, the existence of such a function yields:

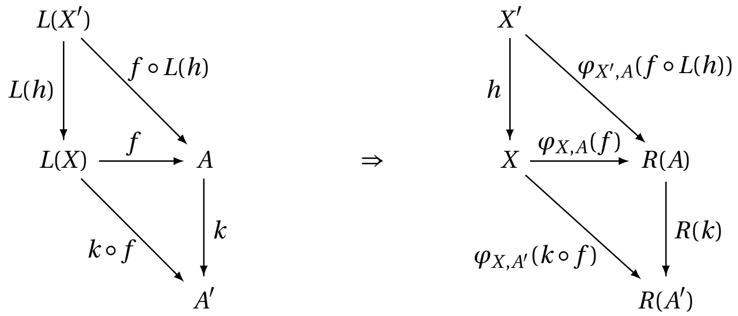
$$\begin{aligned} f : L(X) \to A \quad &\Rightarrow \quad \varphi_{X,Y}(f) : X \to R(A), \\ g : X \to R(A) \quad &\Rightarrow \quad \varphi^{-1}_{X,Y}(g) : L(X) \to A, \end{aligned}$$

for $X \in \mathsf{Ob}(\mathsf{C})$ and $Y \in \mathsf{Ob}(D)$.

For $A, A' \in \mathsf{Ob}(D)$ and $X', X \in \mathsf{Ob}(\mathsf{C})$, let

$$\begin{aligned} f : L(X) \to A, \quad & \quad k : A \to A', \\ g : X \to L(A) \quad & \quad h : X' \to X. \end{aligned}$$

The first two conditions on the naturality of (2.4) are captured by the following requirement:

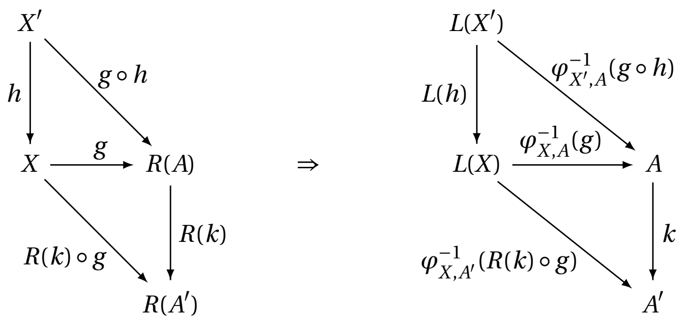


where the implication is that the commutativity of the first diagram implies the commutativity of the second diagram. In particular, this implies that:

$$\varphi_{X,A'}(k \circ f) \quad = \quad R(k) \circ \varphi_{X,A}(f), \tag{2.5}$$

$$\varphi_{X',A}(f \circ L(h)) \quad = \quad \varphi_{X,A}(f) \circ h. \tag{2.6}$$

The second two conditions on the naturality of (2.4) are captured by the following requirement:



In particular, this implies that:

$$\varphi^{-1}_{X',A}(g \circ h) \quad = \quad \varphi^{-1}_{X,A}(g) \circ L(h), \tag{2.7}$$

$$\varphi^{-1}_{X,A'}(R(k) \circ g) \quad = \quad k \circ \varphi^{-1}_{X,A}(g). \tag{2.8}$$

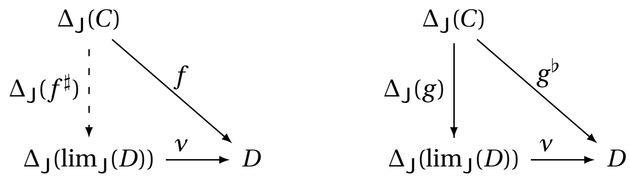
2.2.2 The limit as a right adjoint. The limit yields an adjunction:

$$\Delta_{\mathsf{J}} : \mathsf{C} \rightleftarrows \mathsf{C}^{\mathsf{J}} : \lim_{\mathsf{J}}.$$

That is, C is complete if the constant functor Δ_{J} has a right adjoint for every small category J. The natural bijection

$$\mathsf{Hom}_{\mathsf{C}^{\mathsf{J}}}(\Delta_{\mathsf{J}}(C), D) \cong \mathsf{Hom}_{\mathsf{C}}(C, \lim_{\mathsf{J}}(D)) \tag{2.9}$$

for $C \in \mathsf{Ob}(\mathsf{C})$ and $D \in \mathsf{Ob}(\mathsf{C}^{\mathsf{J}})$ is defined for $f : \Delta_{\mathsf{J}}(C) \xrightarrow{\bullet} D$ and $g : C \to \lim_{J}(D)$ by requiring that there are commuting diagrams:



for ν the universal natural transformation associated with $\lim_{\mathsf{J}}(D)$.

2.3 Colimits in Categories of Hybrid Objects

Since the limit is right adjoint to the constant functor:

$$\Delta_{\mathsf{J}} : \mathsf{C} \rightleftarrows \mathsf{C}^{\mathsf{J}} : \lim_{\mathsf{J}}$$

the natural question to ask is: what is the left adjoint to the constant functor? The answer is that, if the left adjoint exists, it is the dual of the limit: the colimit. Specifically a category is cocomplete if for every small category J, the left adjoint to the constant functor exists, i.e., if there is an adjunction:

$$\mathrm{colim}_{\mathsf{J}} : \mathsf{C}^{\mathsf{J}} \rightleftarrows \mathsf{C} : \Delta_{\mathsf{J}}.$$

We have proven that if C is a complete category, the $\mathsf{Hy}(\mathsf{C})$ is complete. To goal is to prove that:

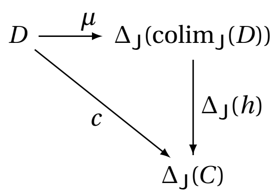
If C is cocomplete, then $\mathsf{Hy}(\mathsf{C})$ is cocomplete.

In order to prove this, we will introduce left Kan extensions.

2.3.1 Colimits. The dual to the limit is the colimit. Let C be a category, J a small category and $D : \mathsf{J} \to \mathsf{C}$ a functor. The colimit of D, if it exists, is an object $\mathrm{colim}_{\mathsf{J}}(D)$ of C together with a *universal* natural transformation:

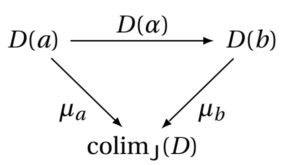
$$\mu : D \stackrel{\cdot}{\to} \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D)).$$

The universality of the colimit is captured by the condition that for any other object C of C with a natural transformation $c : D \stackrel{\cdot}{\to} \Delta_{\mathsf{J}}(C)$ there exists a unique morphism $h : \mathrm{colim}_{\mathsf{J}}(D) \to C$ making the following diagram:

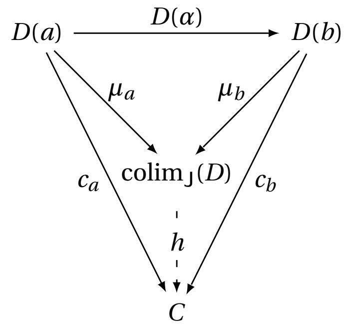


commute.

More explicitly, the definition of the colimit implies that for every $\alpha : a \to b$ in J, the following diagram



commutes. The universality of the colimit implies that the unique morphism $h : \mathrm{colim}_{\mathsf{J}}(D) \to C$ makes the following diagram commute



Definition 2.5. A category C is cocomplete if for every small category J and every functor $D : \mathsf{J} \to \mathsf{C}$, the colimit of D exists.

There is the following useful lemma related to the cocompleteness of a category:

Lemma 2.3. *If C is cocomplete, then it has an initial object.*

Example 2.5. The category of sets, Set, is cocomplete.

2.3.2 Special cases of the colimit. There are many special cases of the colimit that are often of interest. Here we enumerate the most important of these.

Coproducts. The colimit of a functor

$$D : \mathsf{I} \to \mathsf{C},$$

where I is the discrete category obtained from an indexing set I. These are dual to products.

Coequalizers. The colimit of a functor

$$D : (\bullet \rightrightarrows \bullet) \to \mathsf{C}.$$

These are dual to equalizers.

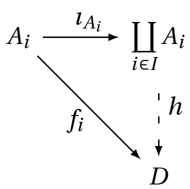
Pushouts. The colimit of a functor

$$D : (\bullet \leftarrow \bullet \to \bullet) \to \mathsf{C}.$$

These are dual to pullbacks.

To understand the implications of these definitions, and because they will be important when discussing model categories, we discuss coproducts and pushouts in more detail.

2.3.3 Coproducts. Let I be a set and $\{A_i\}_{i\in I}$ a set of objects of C. The coproduct of these objects is an object $\coprod_{i\in I} A_i$ in C together with *inclusions* $\iota_{A_i} : A_i \to \coprod_{i\in I} A_i$. It again must satisfy the universal property that for any other object D of C together with morphisms $f_i : A_i \to D$, there exists a unique morphism h making the diagram:



commute. The morphism h is typically denoted by $\langle f_i \rangle_{i\in I}$.

If $\{A_i\}_I$ and $\{B_i\}_I$ are two sets of objects in C and $f_i : A_i \to B_i$ is a collection of morphisms. Then, by the universality of the coproduct, there exists a unique morphism

$$\coprod_{i\in I} f_i := \langle \iota_{B_i} \circ f_i \rangle_{i\in I} : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$$

making the following diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\iota_{A_i}} & \coprod_{i\in I} A_i \\ {\scriptstyle f_i}\downarrow & & \downarrow {\scriptstyle \coprod_{i\in I} f_i := \langle \iota_{B_i}\circ f_i\rangle_{i\in I}} \\ B_i & \xrightarrow{\iota_{B_i}} & \coprod_{i\in I} B_i \end{array}$$

commute.

Example 2.6. The coproduct of a collection of sets is the disjoint union of the sets.

2.3.4 Pushouts. Consider a category C and a diagram of the form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

The pushout of this diagram, if it exists, is an object of $B \amalg_A C$ of C together with two morphisms p and q such that the following *pushout* diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{q} & B \amalg_A C \end{array}$$

commutes. It is universal in the following sense: for any other object D of C with morphisms u and v making the following diagram commute

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u} \\ C & \xrightarrow{v} & D \end{array}$$

there exists a unique morphism $h : B \amalg_A C \to D$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} & \searrow{\scriptstyle u} & \\ C & \xrightarrow{q} & B \amalg_A C & & \\ & \searrow{\scriptstyle v} & & \overset{h}{\dashrightarrow} & D \end{array}$$

Pushouts are very useful conceptual objects when considering D-categories, and more generally hybrid objects, since for every $a \in \mathsf{E}(\mathscr{D})$ for a D-category $\mathscr{D}$, there is a corresponding diagram:

$$\begin{array}{ccc} a & \xrightarrow{\mathsf{s}_a} & b \\ {\scriptstyle \mathsf{t}_a}\downarrow & & \\ c & & \end{array}$$

This indicates, rightly so, that colimits will be of more interest then limits when considering categories of hybrid objects.

2.3.5 The colimit as a functor. If C is a cocomplete category, then the colimit defines a functor

$$\mathrm{colim}_{\mathsf{J}} : \mathsf{C}^{\mathsf{J}} \to \mathsf{C}.$$

We already have defined the object function—it is the colimit of a functor. To define the morphism function, consider a natural transformation $\tau : D \stackrel{\bullet}{\to} D'$. In this case, there are universal natural transformations

$$\begin{array}{rcl} \mu : D & \stackrel{\bullet}{\to} & \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D)) \\ \mu' : D' & \stackrel{\bullet}{\to} & \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D')) \end{array}$$

which yield a natural transformation:

$$\mu' \bullet \tau : D \stackrel{\bullet}{\to} \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D')).$$

This implies there is a unique morphism $\mathrm{colim}_{\mathsf{J}}(D) \to \mathrm{colim}_{\mathsf{J}}(D')$ which is defined to be $\mathrm{colim}_{\mathsf{J}}(\tau)$. That is, there is a commuting diagram:

$$\begin{array}{ccc} D & \xrightarrow{\mu} & \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D)) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(\tau))} \\ D' & \xrightarrow{\mu'} & \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D')) \end{array}$$

Coupling this with the definition of the limit as a functor yields the following commutative diagram:

$$\begin{array}{ccccc}
\Delta_{\mathsf{J}}(\lim_{\mathsf{J}}(D)) & \xrightarrow{\nu} & D & \xrightarrow{\mu} & \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D)) \\
\Delta_{\mathsf{J}}(\lim_{\mathsf{J}}(f)) \big\downarrow & & \big\downarrow f & & \big\downarrow \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(f)) \\
\Delta_{\mathsf{J}}(\lim_{\mathsf{J}}(D')) & \xrightarrow{\nu'} & D' & \xrightarrow{\mu'} & \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D'))
\end{array}$$

2.3.6 The colimit as a left adjoint. The colimit yields an adjunction:

$$\mathrm{colim}_{\mathsf{J}} : \mathsf{C}^{\mathsf{J}} \rightleftarrows \mathsf{C} : \Delta_{\mathsf{J}}. \tag{2.10}$$

That is, C is cocomplete if the constant functor Δ_{J} has a left adjoint for every small category J. The natural bijection

$$\mathsf{Hom}_{\mathsf{C}}(\mathrm{colim}_{\mathsf{J}}(D), C) \cong \mathsf{Hom}_{\mathsf{C}^{\mathsf{J}}}(D, \Delta_{\mathsf{J}}(C)) \tag{2.11}$$

for $D \in \mathsf{Ob}(\mathsf{C}^{\mathsf{J}})$ and $C \in \mathsf{Ob}(\mathsf{C})$ is defined for $f : \mathrm{colim}_{\mathsf{J}}(D) \to C$ and $g : D \overset{\centerdot}{\to} \Delta_{\mathsf{J}}(C)$ by requiring that it make the following diagrams commute

$$\begin{array}{ccc}
D & \xrightarrow{\mu} & \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D)) \\
& {}_{f^{\sharp}}\searrow & \big\downarrow \Delta_{\mathsf{J}}(f) \\
& & \Delta_{\mathsf{J}}(C)
\end{array}
\qquad
\begin{array}{ccc}
D & \xrightarrow{\mu} & \Delta_{\mathsf{J}}(\mathrm{colim}_{\mathsf{J}}(D)) \\
& {}_{g}\searrow & \big\downarrow \Delta_{\mathsf{J}}(g^{\flat}) \\
& & \Delta_{\mathsf{J}}(C)
\end{array}$$

2.3.7 Preservation of colimits. The fact that the colimit is left adjoint to the constant functor allows us to understand the relationship between colimits in different categories based upon adjunctions between these categories. First, recall that a functor $L : \mathsf{C} \to \mathsf{D}$ induces a functor $L_* : \mathsf{C}^{\mathsf{J}} \to \mathsf{D}^{\mathsf{J}}$ between functor categories.

Theorem 2.3. *Let C be a cocomplete category. If the functor $L : \mathsf{C} \to \mathsf{D}$ has a right adjoint, then* $\mathrm{colim}_{\mathsf{J}}(L_*(D))$ *exists and*

$$L(\mathrm{colim}_{\mathsf{J}}(D)) = \mathrm{colim}_{\mathsf{J}}(L_*(D))$$

for all $D : \mathsf{J} \to \mathsf{C}$,

Proof. Dual of the statement given in Theorem 1, page 119, [18]. □

Corollary 2.2. *Let C be a cocomplete category. If D is isomorphic to C, then D is cocomplete.*

2.3.8 Completeness of Dcat. There is an isomorphism of categories:

$$\mathsf{Grph} \cong \mathsf{Set}^{\bullet \rightrightarrows \bullet},$$

which sends a graph $\Gamma = (Q, E)$ to a functor $D : (\bullet \rightrightarrows \bullet) \to \mathsf{Set}$ given by:

$$D(\bullet \rightrightarrows \bullet) = \left(E \underset{\mathsf{tar}}{\overset{\mathsf{sor}}{\rightrightarrows}} Q \right).$$

Utilizing the fact that:

Proposition 2.5. *If* C *is cocomplete, then* C^{J} *is cocomplete for every small category* J.

This implies that Grph is cocomplete. Moreover, it was demonstrated that $\mathsf{Dcat} \cong \mathsf{Grph}$. Therefore, we have the following:

Theorem 2.4. *The categories of D-categories,* Dcat*, is cocomplete. Specifically, for a diagram* $D : \mathsf{J} \to \mathsf{Dcat}$*,* J *small, the colimit is given by:*

$$\mathrm{colim}_{\mathsf{J}}(D) = \mathrm{dcat}(\mathrm{colim}_{\mathsf{J}}(\mathrm{grph}_*(D)))$$

where $\mathrm{colim}_{\mathsf{J}}(\mathrm{grph}_*(D))$ *is the colimit in the category of graphs.*

Note that the same argument can be utilized to show that Dcat is complete. This provides a simpler proof than the "by hand" proof performed in the beginning of this chapter.

We now introduce left Kan extensions; these will be instrumental in proving the cocompletness of $\mathsf{Hy}(\mathsf{C})$ given the cocompletness of C.

2.3.9 Left Kan extensions. Given a morphism of D-categories $\vec{F} : \mathscr{A} \to \mathscr{B}$, the *Left Kan extension* of $\vec{F}$, denoted by:

$$\vec{F}^k : \mathsf{M}^{\mathscr{A}} \to \mathsf{M}^{\mathscr{B}},$$

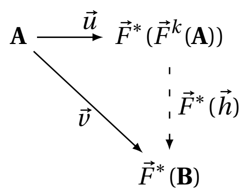
is left adjoint to the pullback functor F^*. Specifically, for a functor $\mathbf{A} : \mathscr{A} \to \mathsf{M}$, there is an associated functor

$$\vec{F}^k(\mathbf{A}) : \mathscr{B} \to \mathsf{M}$$

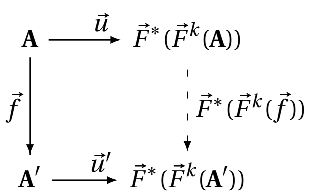
together with a natural transformation

$$\vec{u} : \mathbf{A} \xrightarrow{\cdot} \vec{F}^*(\vec{F}^k(\mathbf{A}))$$

that is universal, i.e., for any other functor $\mathbf{B} : \mathscr{B} \to \mathsf{M}$ and natural transformation $\vec{v} : \mathbf{A} \xrightarrow{\cdot} \vec{F}^*(\mathbf{B})$, there exist a unique $\vec{h} : \vec{F}^k(\mathbf{A}) \xrightarrow{\cdot} \mathbf{B}$ such that the following diagram:



commutes. To see that $\vec{F}^k$ defines a functor, let $\vec{f} : \mathbf{A} \xrightarrow{\cdot} \mathbf{A}'$. Then $\vec{F}^k(\vec{f}) : \vec{F}^k(\mathbf{A}) \xrightarrow{\cdot} \vec{F}^k(\mathbf{A}')$ is the unique morphism making the following diagram commute:



whose existence is asserted by the morphism $\vec{u}' \bullet \vec{f}$.

The definition of left Kan extensions implies that there is an adjunction

$$\vec{F}^k : \mathsf{M}^{\mathscr{A}} \leftrightarrows \mathsf{M}^{\mathscr{B}} : \vec{F}^* \tag{2.12}$$

if C is cocomplete. The bijection of Hom sets:

$$\mathsf{Hom}_{\mathsf{C}^{\mathscr{B}}}(\vec{F}^k(\mathbf{A}), \mathbf{B}) \cong \mathsf{Hom}_{\mathsf{C}^{\mathscr{A}}}(\mathbf{A}, \vec{F}^*(\mathbf{B})) \tag{2.13}$$

is given for:

$$\vec{f} : \vec{F}^k(\mathbf{A}) \stackrel{\cdot}{\to} \mathbf{B}, \qquad \vec{g} : \mathbf{A} \stackrel{\cdot}{\to} \vec{F}^*(\mathbf{B})$$

by defining $\vec{f}^\sharp$ and $\vec{g}^\flat$ as outlined in the following diagrams:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\vec{u}} & \vec{F}^*(\vec{F}^k(\mathbf{A})) \\ & \searrow^{\vec{f}^\sharp} & \downarrow{\scriptstyle \vec{F}^*(\vec{f})} \\ & & \vec{F}^*(\mathbf{B}) \end{array} \qquad \begin{array}{ccc} \mathbf{A} & \xrightarrow{\vec{u}} & \vec{F}^*(\vec{F}^k(\mathbf{A})) \\ & \searrow^{\vec{g}} & \downarrow{\scriptstyle \vec{F}^*(\vec{g}^\flat)} \\ & & \vec{F}^*(\mathbf{B}) \end{array}$$

It is easy to verify that $(\vec{G} \circ \vec{F})^k = \vec{G}^k \circ \vec{F}^k$.

Notation 2.3. At this point, we again use the notation

$$\mathrm{colim}_{\mathsf{J}}^{\mathsf{C}} : \mathsf{C}^{\mathsf{J}} \to \mathsf{C}, \qquad \Delta_{\mathsf{J}}^{\mathsf{C}} : \mathsf{C} \to \mathsf{C}^{\mathsf{J}}$$

for the colimit and constant functors.

2.3.10 Diagrams in categories of hybrid objects (revisited). As in Paragraph 2.1.10, we again denote a diagram in $\mathsf{Hy}(\mathsf{C})$ by $(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{Hy}(\mathsf{C})$. Again, we can without ambiguity write $\mathscr{D}^{\mathsf{J}} : \mathsf{J} \to \mathsf{Dcat}$. Since the category of D-categories is cocomplete, there exists a D-category $\mathrm{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})$ together with a natural transformation:

$$\vec{U} : \mathscr{D}^{\mathsf{J}} \stackrel{\cdot}{\to} \Delta_{\mathsf{J}}^{\mathsf{Dcat}}(\mathrm{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})).$$

Note that for every diagram of the form $\alpha : a \to b$ in J there is a corresponding diagram:

$$\begin{array}{ccc} \mathscr{D}^{\mathsf{J}}(a) & \xrightarrow{\mathscr{D}^{\mathsf{J}}(\alpha)} & \mathscr{D}^{\mathsf{J}}(b) \\ & {\scriptstyle \vec{U}_a} \searrow \quad \swarrow {\scriptstyle \vec{U}_b} & \\ & \mathrm{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}}) & \end{array} \tag{2.14}$$

in Dcat.

For the diagram $(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{Hy}(\mathsf{C})$ in $\mathsf{Hy}(\mathsf{C})$, the colimit of $\mathscr{D}^{\mathsf{J}} : \mathsf{J} \to \mathsf{Dcat}$ yields a functor:

$$\vec{U}^k(\mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{C}^{\mathrm{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}$$

defined for every diagram of the form $\alpha : a \to b$ in J by:

$$\vec{U}^k(\mathbf{D}^{\mathsf{J}})(a) := \vec{U}^k_a(\mathbf{D}^{\mathsf{J}}(a)), \qquad \vec{U}^k(\mathbf{D}^{\mathsf{J}})(\alpha) := \vec{U}^k_b(\mathbf{D}^{\mathsf{J}}(\alpha)^{\flat}),$$

where

$$\mathbf{D}^{\mathsf{J}}(\alpha)^{\flat} : \mathscr{D}^{\mathsf{J}}(\alpha)^k(\mathbf{D}^{\mathsf{J}}(a)) \stackrel{\bullet}{\to} \mathbf{D}^{\mathsf{J}}(b)$$

is obtained from $\mathbf{D}^{\mathsf{J}}(\alpha) : \mathbf{D}^{\mathsf{J}}(a) \stackrel{\bullet}{\to} \mathscr{D}^{\mathsf{J}}(\alpha)^*(\mathbf{D}^{\mathsf{J}}(b))$ via the natural bijection (2.13). Note that $\vec{U}^k(\mathbf{D}^{\mathsf{J}})$ is well-defined because of the commutativity of (2.14).

Using this notation, we can now prove that categories of hybrid objects are cocomplete and give an explicit formula for the colimit of a diagram.

Theorem 2.5. *If C is cocomplete, then $\mathsf{Hy}(\mathsf{C})$ is cocomplete. Specifically, for $(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) : \mathsf{J} \to \mathsf{Hy}(\mathsf{C})$, the colimit is given by:*

$$\mathrm{colim}^{\mathsf{Hy}(\mathsf{C})}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) = \left(\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}}), \mathrm{colim}^{\mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})}}_{\mathsf{J}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right).$$

Proof. The first step is to find the universal natural transformation in $\mathsf{Hy}(\mathsf{C})^{\mathsf{J}}$

$$\mu : (\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) \stackrel{\bullet}{\to} \Delta^{\mathsf{Hy}(\mathsf{C})}_{\mathsf{J}}\left(\mathrm{colim}^{\mathsf{Hy}(\mathsf{C})}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}})\right).$$

There are universal natural transformations

$$\begin{aligned} \vec{U} : \mathscr{D}^{\mathsf{J}} &\stackrel{\bullet}{\to} \Delta^{\mathsf{Dcat}}_{\mathsf{J}}\left(\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})\right) \\ \vec{\mu} : \vec{U}^k(\mathbf{D}^{\mathsf{J}}) &\stackrel{\bullet}{\to} \Delta^{\mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})}}_{\mathsf{J}}\left(\mathrm{colim}^{\mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})}}_{\mathsf{J}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right) \end{aligned}$$

in $\mathsf{Dcat}^{\mathsf{J}}$ and $(\mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})})^{\mathsf{J}}$. Since for $a \in \mathsf{Ob}(\mathsf{J})$,

$$\vec{\mu}_a : \vec{U}^k(\mathbf{D}^{\mathsf{J}})(a) = \vec{U}^k_a(\mathbf{D}^{\mathsf{J}}(a)) \stackrel{\bullet}{\to} \mathrm{colim}^{\mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})}}_{\mathsf{J}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))$$

in $\mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})}$, the natural bijection (2.13) yields a morphism:

$$\vec{\mu}^{\sharp_{\vec{U}_a}}_a : \mathbf{D}^{\mathsf{J}}(a) \stackrel{\bullet}{\to} \vec{U}^*_a\left(\mathrm{colim}^{\mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})}}_{\mathsf{J}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right)$$

in $\mathsf{C}^{\mathscr{D}^{\mathsf{J}}(a)}$; here, the notion "$\sharp_{\vec{U}_a}$" is used to indicate that this morphism is obtained via the natural bijection for the adjunction $\vec{U}^k_a : \mathsf{C}^{\mathscr{D}^{\mathsf{J}}(a)} \rightleftarrows \mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})} : \vec{U}^*_a$. Define the natural transformation μ objectwise, i.e., for all $a \in \mathsf{Ob}(\mathsf{J})$, by:

$$\mu_a = (\vec{U}_a, \vec{\mu}^{\sharp_{\vec{U}_a}}_a) : (\mathscr{D}^{\mathsf{J}}(a), \mathbf{D}^{\mathsf{J}}(a)) \to \mathrm{colim}^{\mathsf{Hy}(\mathsf{C})}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}}, \mathbf{D}^{\mathsf{J}}) = \left(\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}}), \mathrm{colim}^{\mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})}}_{\mathsf{J}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right)$$

Now, we need to verify that μ is in fact a natural transformation. That is, for $\alpha : a \to b$ in J, we need to show that there is a commuting diagram:

$$\begin{array}{ccc} (\mathscr{D}^{\mathsf{J}}(a), \mathbf{D}^{\mathsf{J}}(a)) & \xrightarrow{(\mathscr{D}^{\mathsf{J}}(\alpha), \mathbf{D}^{\mathsf{J}}(\alpha))} & (\mathscr{D}^{\mathsf{J}}(b), \mathbf{D}^{\mathsf{J}}(b)) \\ & {\scriptstyle (\vec{U}_a, \vec{\mu}^{\sharp_{\vec{U}_a}}_a)} \searrow \quad \swarrow {\scriptstyle (\vec{U}_b, \vec{\mu}^{\sharp_{\vec{U}_b}}_b)} & \\ & \left(\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}}), \mathrm{colim}^{\mathsf{C}^{\mathrm{colim}^{\mathsf{Dcat}}_{\mathsf{J}}(\mathscr{D}^{\mathsf{J}})}}_{\mathsf{J}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right) & \end{array}$$

This is equivalent to showing that there is a commuting diagram:

$$\begin{array}{ccc}
\mathbf{D}^{\mathsf{J}}(a) & \xrightarrow{\mathbf{D}^{\mathsf{J}}(\alpha)} & \mathscr{D}^{\mathsf{J}}(\alpha)^*(\mathbf{D}^{\mathsf{J}}(b)) \\
& {\scriptstyle \vec{\mu}_a^{\sharp_{\vec{U}_a}}} \searrow \quad \swarrow {\scriptstyle \mathscr{D}^{\mathsf{J}}(\alpha)^*(\vec{\mu}_b^{\sharp_{\vec{U}_b}})} & \\
& \vec{U}_a^*\left(\operatorname{colim}_{\mathsf{J}}^{\mathsf{C}^{\operatorname{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right) & \\
& \Vert & \\
& \mathscr{D}^{\mathsf{J}}(\alpha)^* \circ \vec{U}_b^*\left(\operatorname{colim}_{\mathsf{J}}^{\mathsf{C}^{\operatorname{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right) &
\end{array} \tag{2.15}$$

by the commutativity of (2.14). To show this, we begin with the commutative diagram

$$\begin{array}{ccc}
\vec{U}_a^k(\mathbf{D}^{\mathsf{J}}(a)) = \vec{U}_b^k \circ \mathscr{D}^{\mathsf{J}}(\alpha)^k(\mathbf{D}^{\mathsf{J}}(a)) & \xrightarrow{\vec{U}_b^k(\mathbf{D}^{\mathsf{J}}(\alpha)^{\flat_{\mathscr{D}^{\mathsf{J}}(\alpha)}})} & \vec{U}_b^k(\mathbf{D}^{\mathsf{J}}(b)) \\
& {\scriptstyle \vec{\mu}_a} \searrow \quad \swarrow {\scriptstyle \vec{\mu}_b} & \\
& \operatorname{colim}_{\mathsf{J}}^{\mathsf{C}^{\operatorname{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}})) &
\end{array}$$

whose existence is implied by the fact that $\vec{\mu}$ is a natural transformation coupled with the definition of $\vec{U}^k(\mathbf{D}^{\mathsf{J}})$. The adjunction $\vec{U}_b^k : \mathsf{C}^{\mathscr{D}^{\mathsf{J}}(b)} \rightleftarrows \mathsf{C}^{\operatorname{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})} : \vec{U}_b^*$ implies that their is a corresponding commuting diagram:

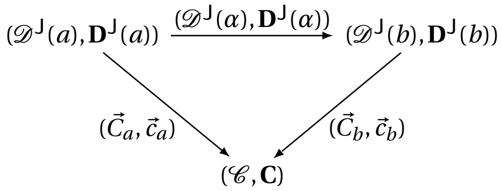
$$\begin{array}{ccc}
\mathscr{D}^{\mathsf{J}}(\alpha)^k(\mathbf{D}^{\mathsf{J}}(a)) & \xrightarrow{\mathbf{D}^{\mathsf{J}}(\alpha)^{\flat_{\mathscr{D}^{\mathsf{J}}(\alpha)}}} & \mathbf{D}^{\mathsf{J}}(b) \\
& {\scriptstyle \vec{\mu}_a^{\sharp_{\vec{U}_b}}} \searrow \quad \swarrow {\scriptstyle \vec{\mu}_b^{\sharp_{\vec{U}_b}}} & \\
& \vec{U}_b^*\left(\operatorname{colim}_{\mathsf{J}}^{\mathsf{C}^{\operatorname{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right) &
\end{array}$$

The adjunction $\mathscr{D}^{\mathsf{J}}(\alpha)^k : \mathsf{C}^{\mathscr{D}^{\mathsf{J}}(a)} \rightleftarrows \mathsf{C}^{\mathscr{D}^{\mathsf{J}}(a)} : \mathscr{D}^{\mathsf{J}}(\alpha)^*$ implies that there is a commuting diagram:

$$\begin{array}{ccc}
\mathbf{D}^{\mathsf{J}}(a) & \xrightarrow{\mathbf{D}^{\mathsf{J}}(\alpha)} & \mathscr{D}^{\mathsf{J}}(\alpha)^*(\mathbf{D}^{\mathsf{J}}(b)) \\
& {\scriptstyle (\vec{\mu}_a^{\sharp_{\vec{U}_b}})^{\sharp_{\mathscr{D}^{\mathsf{J}}(\alpha)}}} \searrow \quad \swarrow {\scriptstyle \mathscr{D}^{\mathsf{J}}(\alpha)^*(\vec{\mu}_b^{\sharp_{\vec{U}_b}})} & \\
& \mathscr{D}^{\mathsf{J}}(\alpha)^* \circ \vec{U}_b^*\left(\operatorname{colim}_{\mathsf{J}}^{\mathsf{C}^{\operatorname{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right) &
\end{array}$$

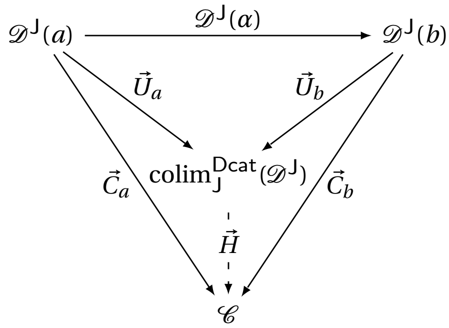
Finally, it follows that $(\vec{\mu}_a^{\sharp_{\vec{U}_b}})^{\sharp_{\mathscr{D}^{\mathsf{J}}(\alpha)}} = \vec{\mu}_a^{\sharp_{\vec{U}_a}}$ by the fact that $\vec{U}_b \circ \mathscr{D}^{\mathsf{J}}(\alpha) = \vec{U}_a$.

To conclude, we need only show the universality of μ. Suppose that there is a hybrid object $(\mathscr{C},\mathbf{C})$ together with a collection of morphisms $(\vec{C}_a,\vec{c}_a):(\mathscr{D}^{\mathsf{J}}(a),\mathbf{D}^{\mathsf{J}}(a))\to(\mathscr{C},\mathbf{C})$ of hybrid objects making the following following diagram



(2.16)

commute for every $\alpha:a\to b$ in J. This yields commuting diagrams and unique morphisms:



(2.17)

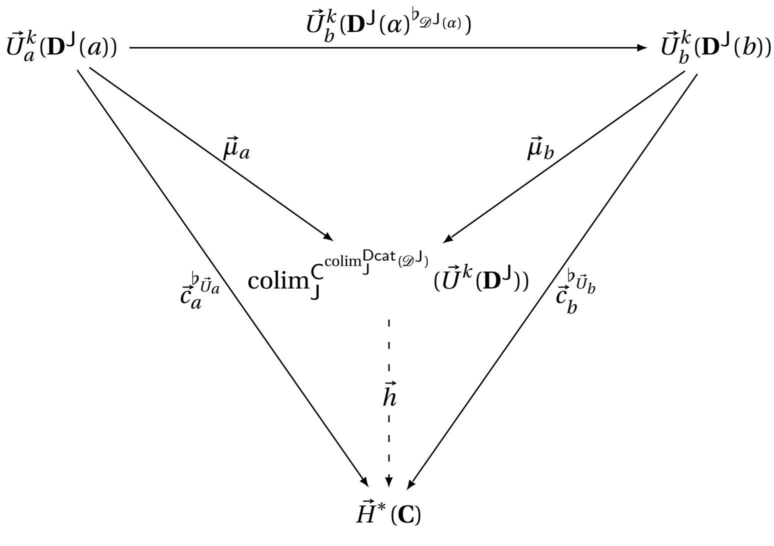
By the universality of the colimit in Dcat. Using the commutativity of this diagram, from the natural transformations:

$$\begin{aligned}\vec{c}_a:\mathbf{D}^{\mathsf{J}}(a) &\stackrel{\cdot}{\to} \vec{C}_a^*(\mathbf{C})=\vec{U}_a^*\circ\vec{H}^*(\mathbf{C})\\ \vec{c}_b:\mathbf{D}^{\mathsf{J}}(b) &\stackrel{\cdot}{\to} \vec{C}_b^*(\mathbf{C})=\vec{U}_b^*\circ\vec{H}^*(\mathbf{C})\end{aligned}$$

we obtain natural transformations:

$$\begin{aligned}\vec{c}_a^{\,\flat_{\vec{U}_a}}:\vec{U}_a^k(\mathbf{D}^{\mathsf{J}}(a)) &\stackrel{\cdot}{\to} \vec{H}^*(\mathbf{C})\\ \vec{c}_b^{\,\flat_{\vec{U}_b}}:\vec{U}_b^k(\mathbf{D}^{\mathsf{J}}(b)) &\stackrel{\cdot}{\to} \vec{H}^*(\mathbf{C})\end{aligned}$$

By the commutativity of (2.16), this implies that there is a commuting diagram:

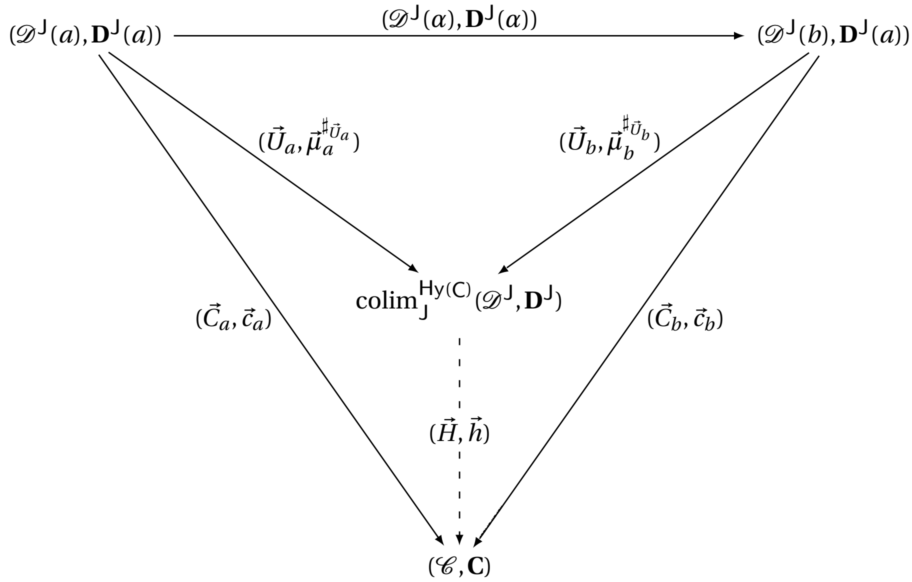


(2.18)

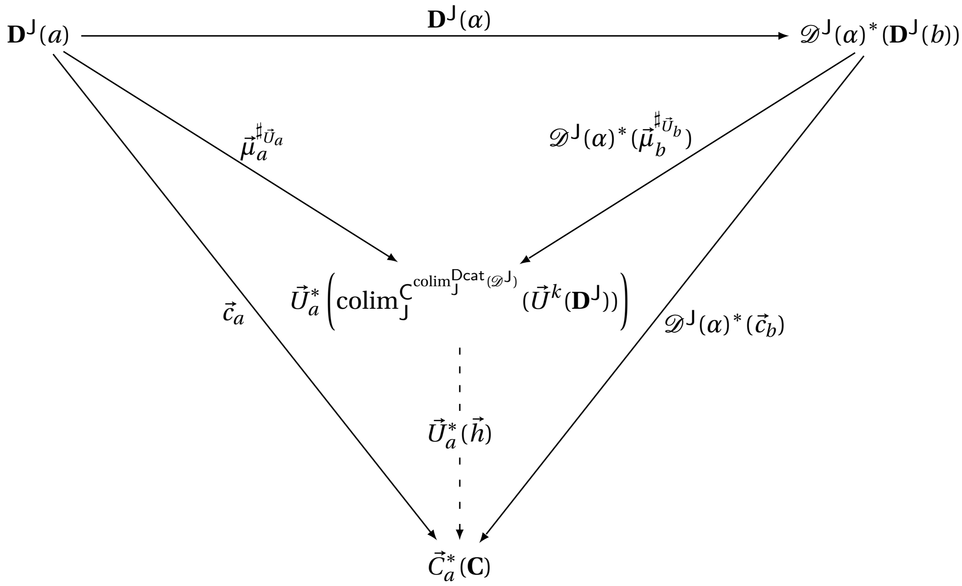
where the natural transformation $\vec{h}$ exists and is unique by the universality of the colimit in $\mathsf{C}^{\mathrm{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}$. Therefore, we have produced a unique morphism of hybrid objects:

$$(\vec{H},\vec{h}) : \mathrm{colim}_{\mathsf{J}}^{\mathsf{Hy(C)}}(\mathscr{D}^{\mathsf{J}},\mathbf{D}^{\mathsf{J}}) \to (\mathscr{C},\mathbf{C})$$

Now, we need only verify that for this morphism, the following diagram:



commutes. To do this, we need only show that the diagram:



commutes. We have already demonstrated that the inner triangle of this diagram commutes and the outer triangle commutes by definition. Therefore, we need only show that:

$$\vec{c}_a = \vec{U}_a^*(\vec{h}) \bullet \vec{\mu}_a^{\sharp\vec{U}_a}, \qquad \mathscr{D}^{\mathsf{J}}(\alpha)^*(\vec{c}_b) = \vec{U}_a^*(\vec{h}) \bullet \mathscr{D}^{\mathsf{J}}(\alpha)^*(\vec{\mu}_b^{\sharp\vec{U}_b}). \tag{2.19}$$

The equality on the left follows from the naturality of the adjunction $\vec{U}_a^k : \mathsf{C}^{\mathscr{D}^{\mathsf{J}}(a)} \rightleftarrows \mathsf{C}^{\mathrm{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})} : \vec{U}_a^*$ coupled with the commutativity of (2.18). To verify that the equality on the right holds, consider the following diagram:

$$\begin{array}{ccc}
\mathbf{D}^{\mathsf{J}}(b) & \xrightarrow{\vec{\mu}_b^{\sharp \vec{U}_b}} & \vec{U}_b^*\left(\mathrm{colim}_{\mathsf{J}}^{\mathsf{C}^{\mathrm{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})}}(\vec{U}^k(\mathbf{D}^{\mathsf{J}}))\right) \\
& \searrow^{\vec{c}_b} & \downarrow{\scriptstyle \vec{U}_b^*(\vec{h})} \\
& & \vec{C}_b^*(\mathbf{C})
\end{array}$$

which commutes because of the naturality of $\vec{U}_b^k : \mathsf{C}^{\mathscr{D}^{\mathsf{J}}(b)} \rightleftarrows \mathsf{C}^{\mathrm{colim}_{\mathsf{J}}^{\mathsf{Dcat}}(\mathscr{D}^{\mathsf{J}})} : \vec{U}_b^*$ coupled with the commutativity of (2.18). Moreover, applying $\mathscr{D}^{\mathsf{J}}(\alpha)^*$ to this diagram again yields a commuting diagram which is just equality on the right of (2.19) by the commutativity of (2.17). □

2.3.11 Change of D-category. Let $\mathbf{B} : \mathscr{B} \to \mathsf{C}$ and

$$\mu : \mathbf{B} \stackrel{\cdot}{\to} \Delta_{\mathscr{B}}(\mathrm{colim}_{\mathscr{B}}(\mathbf{B}))$$

be the colimit of this diagram. For a morphism of D-categories $\vec{F} : \mathscr{A} \to \mathscr{B}$, we obtain a natural transformation:

$$\vec{F}^*(\mu) : \vec{F}^*(\mathbf{B}) \stackrel{\cdot}{\to} \vec{F}^*(\Delta_{\mathscr{B}}(\mathrm{colim}_{\mathscr{B}}(\mathbf{B}))) = \Delta_{\mathscr{A}}(\mathrm{colim}_{\mathscr{B}}(\mathbf{B}))$$

in $\mathsf{C}^{\mathscr{A}}$. This implies that there is a unique morphism

$$\mathrm{colim}_{\vec{F}} := \vec{F}^*(\mu)^\flat : \mathrm{colim}_{\mathscr{A}}(\vec{F}^*(\mathbf{B})) \to \mathrm{colim}_{\mathscr{B}}(\mathbf{B})$$

which makes the following diagram

$$\begin{array}{ccc}
F^*(\mathbf{B}) & \xrightarrow{\mu'} & \Delta_{\mathscr{A}}(\mathrm{colim}_{\mathscr{A}}(\vec{F}^*(\mathbf{B}))) \\
& \searrow^{\vec{F}^*(\mu)} & \downarrow{\scriptstyle \mathrm{colim}_{\vec{F}}} \\
& & \Delta_{\mathscr{A}}(\mathrm{colim}_{\mathscr{B}}(\mathbf{B}))
\end{array}$$

commute.

2.3.12 Hybrid colimits. Using the colimit, we can construct a functor from $\mathsf{Hy}(\mathsf{C})$ to C. Define the functor, also denoted by colim but *without* the subscript,

$$\mathrm{colim} : \mathsf{Hy}(\mathsf{C}) \to \mathsf{C}.$$

on objects $(\mathscr{A}, \mathbf{A})$ of $\mathsf{Hy}(\mathsf{C})$ by

$$\mathrm{colim}(\mathscr{A}, \mathbf{A}) = \mathrm{colim}_{\mathscr{A}}(\mathbf{A}).$$

On morphisms $(\vec{F}, \vec{f}) : (\mathscr{A}, \mathbf{A}) \to (\mathscr{B}, \mathbf{B})$, where again $\vec{f} : \mathbf{A} \stackrel{\cdot}{\to} \vec{F}^*(\mathbf{B})$, we obtain a morphism

$$\mathrm{colim}(\vec{F}, \vec{f}) : \mathrm{colim}_{\mathscr{A}}(\mathbf{A}) \to \mathrm{colim}_{\mathscr{B}}(\mathbf{B}),$$

by defining it to be the unique morphism making the following diagram

$$\begin{array}{ccc} \mathrm{colim}_{\mathscr{A}}(\mathbf{A}) & \xrightarrow{\mathrm{colim}_{\mathscr{A}}(\vec{f})} & \mathrm{colim}_{\mathscr{A}}(\vec{F}^*(\mathbf{B})) \\ & \searrow^{\mathrm{colim}(\vec{F},\vec{f})} & \downarrow{\scriptstyle \mathrm{colim}_{\vec{F}}} \\ & & \mathrm{colim}_{\mathscr{B}}(\mathbf{B}) \end{array}$$

commute.

There is a canonical projection functor:

$$\Pi : \mathsf{Hy(C)} \to \mathsf{Dcat}$$

given on objects by $\Pi(\mathscr{A},\mathbf{A}) = \mathbf{A}$ and on morphisms by $\Pi(\vec{F},\vec{f}) = \vec{F}$. Coupling this functor with the colim functor yields a functor, termed the *hybrid colimit*, and given by:

$$\mathrm{hycolim} := (\Pi, \mathrm{colim}) : \mathsf{Hy(C)} \to \mathsf{Dcat} \times \mathsf{C}.$$

In fact, like the colimit functor, this functor is left adjoint to the *hybrid constant functor*, given by:

$$\Delta^{\mathrm{hy}} := (\,-\,, \Delta_{\,-}\,(\,\cdot\,)) : \mathsf{Dcat} \times \mathsf{C} \to \mathsf{Hy(C)}.$$

That is, for $\mathscr{A} \in \mathsf{Ob(Dcat)}$ and $A \in \mathsf{Ob(C)}$, the corresponding object in $\mathsf{Hy(C)}$ is given by $(\mathscr{A}, \Delta_{\mathscr{A}}(A))$.

These functors yield the analogy of the colimit adjunction for categories of hybrid objects.

Theorem 2.6. *If C is cocomplete, then the pair of functors:*

$$\mathrm{hycolim} = (\Pi, \mathrm{colim}) : \mathsf{Hy(C)} \rightleftarrows \mathsf{Dcat} \times \mathsf{C} : (\,-\,, \Delta_{\,-}\,(\,\cdot\,)) = \Delta^{\mathrm{hy}}$$

form an adjoint pair.

2.3.13 The natural bijection. In order to construct the natural bijection for this adjunction, we use the natural bijection (2.11) for the adjunction (2.10).

Denote the natural bijection for the adjunction given in Theorem 2.6 by:

$$\mathsf{Hom}_{\mathsf{Dcat}\times\mathsf{C}}((\Pi,\mathrm{colim})(\mathscr{X},\mathbf{X}),(\mathscr{A},A)) \cong \mathsf{Hom}_{\mathsf{Hy(C)}}((\mathscr{X},\mathbf{X}),(\mathscr{A},\Delta_{\mathscr{A}}(A))) \tag{2.20}$$

and define it for

$$\begin{aligned} (\vec{F}, f) : (\Pi,\mathrm{colim})(\mathscr{X},\mathbf{X}) &\to (\mathscr{A}, A), \\ (\vec{G},\vec{g}) : (\mathscr{X},\mathbf{X}) &\to (\mathscr{A},\Delta_{\mathscr{A}}(A)), \end{aligned}$$

where, again,

$$\begin{aligned} &\vec{F} : \mathscr{X} \to \mathscr{A}, \qquad f : \mathrm{colim}(\mathscr{X},\mathbf{X}) = \mathrm{colim}_{\mathscr{X}}(\mathbf{X}) \to A, \\ &\vec{G} : \mathscr{X} \to \mathscr{A}, \qquad \vec{g} : \mathbf{X} \stackrel{\centerdot}{\to} \vec{G}^*(\Delta_{\mathscr{X}}(A)) = \Delta_{\mathscr{A}}(A), \end{aligned}$$

by

$$(\vec{F}, f)^{\sharp} := (\vec{F}, f^{\sharp}), \qquad (\vec{G}, \vec{g})^{\flat} := (\vec{G}, \vec{g}^{\flat})$$

where the "$\sharp$" and "$\flat$" operations on the right are with respect to the natural bijection given in (2.11) which is well-defined since:

$$f^{\sharp} : \mathscr{X} \stackrel{\cdot}{\to} \Delta_{\mathscr{X}}(A) = \vec{F}^{*}(\Delta_{\mathscr{A}}(A)).$$

The construction of the bijection (2.20) makes the proof of Theorem 2.6 a simple matter.

2.4 Adjunctions between Categories of Hybrid Objects

The goal of this section is to prove the existence of *adjunctions between categories of hybrid objects* given an adjunction on the target categories. That is, for an adjunction $L : \mathsf{C} \rightleftarrows \mathsf{D} : R$ there is an adjunction

$$\mathsf{Hy}(L) : \mathsf{Hy}(\mathsf{C}) \rightleftarrows \mathsf{Hy}(\mathsf{D}) : \mathsf{Hy}(R).$$

Before proving the existence of adjunctions between categories of hybrid objects, recall that there is the following easy result ([18]).

Lemma 2.4. *If $L : \mathsf{C} \rightleftarrows \mathsf{D} : R$ is an adjunction, then for a D-category $\mathscr{D}$,*

$$L_* : \mathsf{C}^{\mathscr{D}} \rightleftarrows \mathsf{D}^{\mathscr{D}} : R_*$$

is an adjunction.

2.4.1 The natural bijection. Consider the natural bijection

$$\varphi_{X,A} : \mathsf{Hom}_{\mathsf{D}}(L(X), A) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{C}}(X, R(A)) \tag{2.21}$$

given by the adjunction $L : \mathsf{C} \rightleftarrows \mathsf{D} : R$. From this we obtain a natural bijection:

$$\vec{\varphi}_{\mathbf{X},\mathbf{A}} : \mathsf{Hom}_{\mathsf{D}}(L_*(\mathbf{X}), \mathbf{A}) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{C}}(\mathbf{X}, R_*(\mathbf{A})), \tag{2.22}$$

where $\mathbf{X} : \mathscr{D} \to \mathsf{C}$ and $\mathbf{A} : \mathscr{D} \to \mathsf{D}$. For a natural transformation: $\vec{f} : L_*(\mathbf{X}) \stackrel{\cdot}{\to} \mathbf{A}$, the natural transformation:

$$\vec{\varphi}_{\mathbf{X},\mathbf{A}}(\vec{f}) : \mathbf{X} \stackrel{\cdot}{\to} R_*(\mathbf{A})$$

is given objectwise by:

$$\vec{\varphi}_{\mathbf{X},\mathbf{A}}(\vec{f})_a := \varphi_{\mathbf{X}_a,\mathbf{A}_a}(\vec{f}_a) : \mathbf{X}_a \to R(\mathbf{A}_a) = R_*(\mathbf{A})_a.$$

It is easy to see that $\vec{\varphi}$ is a bijection since, for $\vec{g} : \mathbf{X} \to R_*(\mathbf{A})$, it has inverse

$$\vec{\varphi}^{-1}_{\mathbf{X},\mathbf{A}}(\vec{g}) : L_*(\mathbf{X}) \to \mathbf{A}$$

defined objectwise by $\vec{\varphi}^{-1}_{\mathbf{X},\mathbf{A}}(\vec{g})_a = \varphi^{-1}_{\mathbf{X}_a,\mathbf{A}_a}(\vec{g}_a)$. Similarly, the naturality of $\vec{\varphi}$ follows from the naturality of φ.

We are now in a position to prove:

Theorem 2.7. *If $L : \mathsf{C} \rightleftarrows \mathsf{D} : R$ is an adjunction, then*

$$\mathsf{Hy}(L) : \mathsf{Hy}(\mathsf{C}) \rightleftarrows \mathsf{Hy}(\mathsf{D}) : \mathsf{Hy}(R)$$

is an adjunction.

2.4.2 The natural bijection. In order to prove this theorem we produce, explicitly, the natural bijection associated to the adjunction between categories of hybrid objects based on the natural bijection for the corresponding "non-hybrid" adjunction. We begin with the natural bijection (2.22), from which we obtain a natural bijection

$$\begin{array}{c} \mathsf{Hom}_{\mathsf{Hy(D)}}(\mathsf{Hy}(L)(\mathscr{X},\mathbf{X}),(\mathscr{A},\mathbf{A})) \\ \Phi_{(\mathscr{X},\mathbf{X}),(\mathscr{A},\mathbf{A})} \big\downarrow \wr \\ \mathsf{Hom}_{\mathsf{Hy(C)}}((\mathscr{X},\mathbf{X}),\mathsf{Hy}(R)(\mathscr{A},\mathbf{A})) \end{array} \tag{2.23}$$

for $(\mathscr{X},\mathbf{X}) \in \mathsf{Ob}(\mathsf{Hy(C)})$ and $(\mathscr{A},\mathbf{A}) \in \mathsf{Ob}(\mathsf{Hy(D)})$. The function Φ is defined using the function $\vec{\varphi}$. Explicitly, consider a hybrid morphism:

$$(\vec{F},\vec{f}) : \mathsf{Hy}(L)(\mathscr{X},\mathbf{X}) = (\mathscr{X}, L_*(\mathbf{X})) \to (\mathscr{A},\mathbf{A})$$

that is:

$$\vec{F} : \mathscr{X} \to \mathscr{A}, \qquad \vec{f} : L_*(\mathbf{X}) \stackrel{\cdot}{\to} \vec{F}^*(\mathbf{A}).$$

Using the bijection $\vec{\varphi}$ we obtain a natural transformation:

$$\vec{\varphi}_{\mathbf{X},\vec{F}^*(\mathbf{A})}(\vec{f}) : \mathbf{X} \stackrel{\cdot}{\to} R_*(\vec{F}^*(\mathbf{A})) = \vec{F}^*(R_*(\mathbf{A})).$$

It follows that:

$$\Phi_{(\mathscr{X},\mathbf{X}),(\mathscr{A},\mathbf{A})}(\vec{F},\vec{f}) := (\vec{F},\vec{\varphi}_{\mathbf{X},\vec{F}^*(\mathbf{A})}(\vec{f}))$$

is the desired morphism, i.e.,

$$\Phi_{(\mathscr{X},\mathbf{X}),(\mathscr{A},\mathbf{A})}(\vec{F},\vec{f}) : (\mathscr{X},\mathbf{X}) \to \mathsf{Hy}(R)(\mathscr{A},\mathbf{A}) = (\mathscr{A}, R_*(\mathbf{A})).$$

In a similar manner, for

$$(\vec{G},\vec{g}) : (\mathscr{X},\mathbf{X}) \to \mathsf{Hy}(R)(\mathscr{A},\mathbf{A})$$

define

$$\Phi^{-1}_{(\mathscr{X},\mathbf{X}),(\mathscr{A},\mathbf{A})}(\vec{G},\vec{g}) := (\vec{G},\vec{\varphi}^{-1}_{\mathbf{X},\vec{G}^*(\mathbf{A})}(\vec{g})).$$

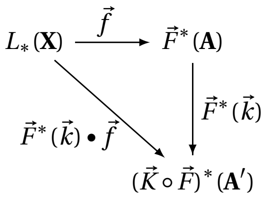
It can be readily verified that Φ and Φ^{-1} are inverses of one another.

Proof. Let, for $(\mathscr{X},\mathbf{X}),(\mathscr{X}',\mathbf{X}') \in \mathsf{Ob}(\mathsf{Hy(C)})$ and $(\mathscr{A},\mathbf{A}),(\mathscr{A}',\mathbf{A}') \in \mathsf{Ob}(\mathsf{Hy(D)})$,

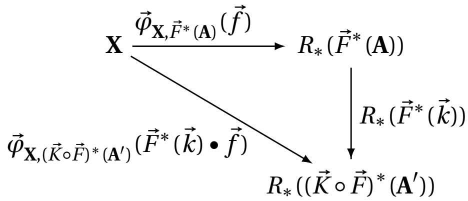
$$\begin{array}{lll} (\vec{F},\vec{f}) : \mathsf{Hy}(L)(\mathscr{X},\mathbf{X}) \to (\mathscr{A},\mathbf{A}), & (\vec{K},\vec{k}) : (\mathscr{A},\mathbf{A}) \to (\mathscr{A}',\mathbf{A}'), & \in \mathsf{Hy(D)} \\ (\vec{G},\vec{g}) : (\mathscr{X},\mathbf{X}) \to \mathsf{Hy}(R)(\mathscr{A},\mathbf{A}) & (\vec{H},\vec{h}) : (\mathscr{X}',\mathbf{X}') \to (\mathscr{X},\mathbf{X}). & \in \mathsf{Hy(C)} \end{array}$$

We will verify the first two conditions on the naturality of Φ, i.e., (2.5) and (2.5). One verifies the last two conditions, (2.7) and (2.8), in an analogous way.

Condition 1 (2.5): For the diagram



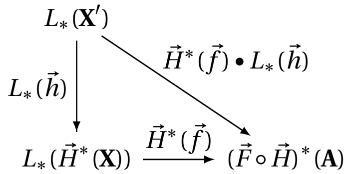
in $\mathsf{D}^{\mathscr{X}}$, the naturality of $\vec{\varphi}$ implies that the following diagram



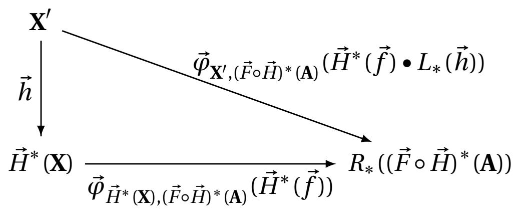
in $\mathsf{C}^{\mathscr{X}}$ commutes. Using this, by direct calculation we have:

$$
\begin{aligned}
\Phi_{(\mathscr{X},\mathbf{X}),(\mathscr{A}',\mathbf{A}')}((\vec{K},\vec{k}) \circ (\vec{F},\vec{f})) &= \Phi_{(\mathscr{X},\mathbf{X}),(\mathscr{A}',\mathbf{A}')}(\vec{K}\circ\vec{F}, \vec{F}^*(\vec{k}) \bullet \vec{f}) \\
&= (\vec{K}\circ\vec{F}, \vec{\varphi}_{\mathbf{X},(\vec{K}\circ\vec{F})^*(\mathbf{A}')}(\vec{F}^*(\vec{k}) \bullet \vec{f})) \\
&= (\vec{K}\circ\vec{F}, R_*(\vec{F}^*(\vec{k})) \bullet \vec{\varphi}_{\mathbf{X},\vec{F}^*(\mathbf{A})}(\vec{f})) \\
&= (\vec{K}\circ\vec{F}, \vec{F}^*(R_*(\vec{k})) \bullet \vec{\varphi}_{\mathbf{X},\vec{F}^*(\mathbf{A})}(\vec{f})) \\
&= (\vec{K}, R_*(\vec{k})) \circ (\vec{F}, \vec{\varphi}_{\mathbf{X},\vec{F}^*(\mathbf{A})}(\vec{f})) \\
&= \mathsf{Hy}(R)(\vec{K},\vec{k}) \circ \Phi_{(\mathscr{X},\mathbf{X}),(\mathscr{A},\mathbf{A})}(\vec{F},\vec{f}).
\end{aligned}
$$

Condition 2 (2.6): For the diagram



in $\mathsf{D}^{\mathscr{X}'}$, by the naturality of $\vec{\varphi}$, we have the following commuting diagram



in $\mathsf{C}^{\mathscr{X}'}$. It addition:

$$
\vec{\varphi}_{\vec{H}^*(\mathbf{X}),(\vec{F}\circ\vec{H})^*(\mathbf{A})}(\vec{H}^*(\vec{f})) = \vec{H}^*(\vec{\varphi}_{\mathbf{X},\vec{F}^*(\mathbf{A})}(\vec{f})).
$$

Therefore, a direct calculation thus yields:

$$\begin{aligned}
\Phi_{(\mathscr{X}',\mathbf{X}'),(\mathscr{A},\mathbf{A}')}((\vec{F},\vec{f})\bullet \mathsf{Hy}(L)(\vec{H},\vec{h})) &= \Phi_{(\mathscr{X}',\mathbf{X}'),(\mathscr{A},\mathbf{A}')}(\vec{F}\circ\vec{H},\vec{H}^*(\vec{f})\bullet L_*(\vec{h}))\\
&= (\vec{F}\circ\vec{H},\vec{\varphi}_{\mathbf{X}',(\vec{F}\circ\vec{H})^*(\mathbf{A})}(\vec{H}^*(\vec{f})\bullet L_*(\vec{h})))\\
&= (\vec{F}\circ\vec{H},\vec{\varphi}_{\vec{H}^*(\mathbf{X}),(\vec{F}\circ\vec{H})^*(\mathbf{A})}(\vec{H}^*(\vec{f}))\bullet\vec{h})\\
&= (\vec{F}\circ\vec{H},\vec{H}^*(\vec{\varphi}_{\mathbf{X},\vec{F}^*(\mathbf{A})}(\vec{f}))\bullet\vec{h})\\
&= (\vec{F},\vec{\varphi}_{\mathbf{X},\vec{F}^*(\mathbf{A})}(\vec{f}))\bullet(\vec{H},\vec{h})\\
&= \Phi_{(\mathscr{X},\mathbf{X}),(\mathscr{A},\mathbf{A})}(\vec{F},\vec{f})\bullet(\vec{H},\vec{h}).
\end{aligned}$$

□

2.5 Categories of Hybrid Objects as Fibered Categories

It is important to understand how categories of hybrid objects fit within the broad context of category theory. In this section, we demonstrate that categories of hybrid objects are fibered categories. We refer the reader to [4] for more on categories of this form. An additional motivation for understanding how categories of hybrid objects fit within the framework of fibered categories is that the are preexisting results on the model structure of these categories [1].

2.5.1 Fiber bundles. Fibered categories generalize the notion of fiber bundles in topology. That is, one begins with a topological space, B, termed the *base space*, a topological space E termed the *total space*, and a continuous surjection:

$$p : E \to B$$

termed the *projection map*. In addition, for each point $x \in B$, there is a set Y_x, termed the *fiber over* x, and given by:

$$Y_x = p^{-1}(x).$$

In addition, there are some additional "consistency conditions" that this data is required to satisfy for it to be considered a fiber bundle.

2.5.2 Fibrations. The notion of a fibration in the framework of category theory follows along much the same lines—except, topological spaces are replaced by categories and maps are replaced by functors.

Specifically, one begins with a category, B, termed the *base category*, a category E, termed the *total category* and a functor:

$$P : \mathsf{E} \to \mathsf{B}$$

termed the *projection functor*. In addition, there is the notion of a *fiber over I* for $I \in \mathsf{Ob}(\mathsf{B})$; this is a category

$$\mathsf{E}_I = P^{-1}(I).$$

By this, formally, we mean that E_I is a subcategory of E satisfying, for $X, Y \in \mathsf{Ob}(\mathsf{E})$,

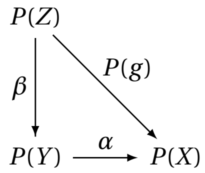
$$\begin{array}{lcl} X \in \mathsf{Ob}(\mathsf{E}_I) & \text{if} & P(X) = I \\ f : X \to Y \in \mathsf{E}_I & \text{if} & P(f) = \mathrm{id}_I \end{array}$$

In addition, some consistency conditions are required; these are manifested through the notion of a cartesian morphism.

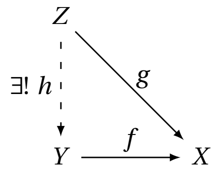
Definition 2.6. Let $P : \mathsf{E} \to \mathsf{B}$ be a functor and $\alpha : J \to I$ a morphism in B. A morphism $f : Y \to X$ in E is *cartesian over α* if:

(i) $P(f) = \alpha$,

(ii) For every morphism $g : Z \to X$ such that there exists a $\beta : P(Z) \to P(X)$ making the following diagram commute:



there exists a unique h making the following diagram



commute.

Definition 2.7. A functor $P : \mathsf{E} \to \mathsf{B}$ is a *fibration* if for every morphism $\alpha : J \to I$ and every $X \in \mathsf{Ob}(\mathsf{E}_I)$ there exists a morphism $f : Y \to X$ that is cartesian over α. In this case, E is *fibered over* B.

2.5.3 Categories of hybrid objects as fibered categories. It will be seen that $\mathsf{Hy}(\mathsf{C})$ has the structure of a fibered category with the fibration given by the projection functor (see Paragraph 2.3.12)

$$\Pi : \mathsf{Hy}(\mathsf{C}) \to \mathsf{Dcat}.$$

For a D-category $\mathscr{X}$, the fiber of $\mathsf{Hy}(\mathsf{C})$ at $\mathscr{X}$ is characterized by:

$$\mathsf{Hy}(\mathsf{C})_{\mathscr{X}} \cong \mathsf{C}^{\mathscr{X}}$$

since the objects in $\mathsf{Hy}(\mathsf{C})_{\mathscr{X}}$ are of the form $(\mathscr{X}, \mathbf{X})$ and the morphisms in this fiber are of the form $(\vec{\mathrm{Id}}_{\mathscr{X}}, \vec{f}) : (\mathscr{X}, \mathbf{X}) \to (\mathscr{X}, \mathbf{X}')$, so $\vec{f} : \mathbf{X} \dot{\to} \mathbf{X}'$ and the isomorphism with $\mathsf{C}^{\mathscr{X}}$ is given by projecting onto the second component:

$$(\mathscr{X}, \mathbf{X}) \mapsto \mathbf{X}, \qquad (\vec{\mathrm{Id}}_{\mathscr{X}}, \vec{f}) \mapsto \vec{f}.$$

We now describe cartesian morphisms in $\mathsf{Hy}(\mathsf{C})$.

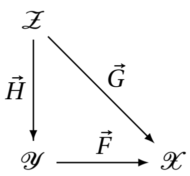
2.5.4 Cartesian morphisms. We demonstrate that there are cartesian morphisms. Let $\vec{F} : \mathscr{Y} \to \mathscr{X}$. An object in the fiber $\mathsf{Hy}(\mathsf{C})_{\mathscr{X}}$ is of the form $(\mathscr{X}, \mathbf{X})$, from which we obtain the object

$$(\mathscr{X}, \mathbf{X})_{\vec{F}} := (\mathscr{Y}, \vec{F}^*(\mathbf{X}))$$

in $\mathsf{Hy}(\mathsf{C})_{\mathscr{Y}}$. This yields a canonical morphism:

$$\vec{F}_{(\mathscr{X},\mathbf{X})} := (\vec{F}, \vec{F}^*(\vec{\mathrm{id}}_{\mathbf{X}})) : (\mathscr{X}, \mathbf{X})_{\vec{F}} \to (\mathscr{X}, \mathbf{X}) \tag{2.24}$$

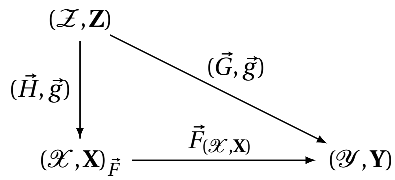
which is a cartesian morphism over $\vec{F}$. It clearly satisfies (i) in Definition 2.6. To verify (ii), consider an object $(\mathscr{Z}, \mathbf{Z})$ and a morphism $(\vec{G}, \vec{g}) : (\mathscr{Z}, \mathbf{Z}) \to (\mathscr{X}, \mathbf{X})$ such that there exists a commuting diagram



in Dcat. This yields a morphism

$$(\vec{H}, \vec{g}) : (\mathscr{Z}, \mathbf{Z}) \to (\mathscr{X}, \mathbf{X})_{\vec{F}},$$

which is unique by construction and makes the following diagram:



commute. That is, we have demonstrated:

Proposition 2.6. $\Pi : \mathsf{Hy}(\mathsf{C}) \to \mathsf{Dcat}$ *is a fibration, i.e.,* $\mathsf{Hy}(\mathsf{C})$ *is fibered over* Dcat.

The reason for establishing the relationship between categories of hybrid objects and fibered categories is that we can leverage results relating to fibered categories. For example, we know that for a small category J, the functor:

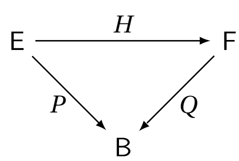
$$\Pi_* : \mathsf{Hy}(\mathsf{C})^{\mathsf{J}} \to \mathsf{Dcat}^{\mathsf{J}}$$

is a fibration.

2.5.5 Cartesian functors. At this point, the natural question to ask is: what are "morphisms" of cartesian categories? This question is addressed through the following definition.

Definition 2.8. Let $P : \mathsf{E} \to \mathsf{B}$ and $Q : \mathsf{F} \to \mathsf{B}$ be two fibrations. A functor $H : \mathsf{E} \to \mathsf{F}$ is *cartesian* if

(i) The following diagram commutes:



(ii) If f is a cartesian morphism for $P : \mathsf{E} \to \mathsf{B}$, then $H(f)$ is a cartesian morphism for $Q : \mathsf{F} \to \mathsf{B}$.

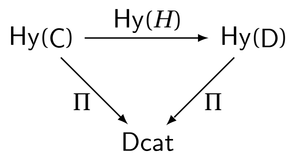
2.5.6 Functors between categories of hybrid objects as cartesian functors. An important observation regarding categories of hybrid objects is that functors between categories of hybrid objects are *always* cartesian.

Proposition 2.7. *Let* $H : \mathsf{C} \to \mathsf{D}$ *be a functor. The corresponding functor:*

$$\mathsf{Hy}(H) : \mathsf{Hy}(\mathsf{C}) \to \mathsf{Hy}(\mathsf{D})$$

is cartesian.

Proof. To verify (i), we note that the diagram[1]



trivially commutes. Now, we know from the construction given in Paragraph 2.5.3 that, for $\mathsf{Hy}(\mathsf{C})$, cartesian morphisms over $\vec{F} : \mathscr{Y} \to \mathscr{X}$ have the form:

$$\vec{F}_{(\mathscr{X},\mathbf{X})} : (\mathscr{X},\mathbf{X})_{\vec{F}} \to (\mathscr{X},\mathbf{X}).$$

Applying $\mathsf{Hy}(H)$ to $(\mathscr{X},\mathbf{X})_{\vec{F}}$ and $\vec{F}_{(\mathscr{X},\mathbf{X})}$ yields:

$$\begin{array}{rcl}
\mathsf{Hy}(H)((\mathscr{X},\mathbf{X})_{\vec{F}}) & = & (\mathscr{Y}, H_*(\vec{F}^*(\mathbf{X}))) \\
& = & (\mathscr{Y}, \vec{F}^*(H_*(\mathbf{X}))) \\
& = & (\mathscr{X}, H_*(\mathbf{X}))_{\vec{F}} \\
\mathsf{Hy}(H)(\vec{F}_{(\mathscr{X},\mathbf{X})}) & = & \mathsf{Hy}(H)(\vec{F}, \vec{F}^*(\vec{\mathrm{id}}_{\mathbf{X}})) \\
& = & (\vec{F}, H_*(\vec{F}^*(\vec{\mathrm{id}}_{\mathbf{X}}))) \\
& = & (\vec{F}, \vec{F}^*(H_*(\vec{\mathrm{id}}_{\mathbf{X}}))) \\
& = & (\vec{F}, \vec{F}^*(\vec{\mathrm{id}}_{H_*(\mathbf{X})})) \\
& = & \vec{F}_{(\mathscr{X},H_*(\mathbf{X}))}.
\end{array}$$

That is

$$\begin{array}{c}
\mathsf{Hy}(H)(\vec{F}_{(\mathscr{X},\mathbf{X})}) : \mathsf{Hy}(H)((\mathscr{X},\mathbf{X})_{\vec{F}}) \to \mathsf{Hy}(H)(\mathscr{X},\mathbf{X}) \\
\shortparallel \\
\vec{F}_{(\mathscr{X},H_*(\mathbf{X}))} : (\mathscr{X}, H_*(\mathbf{X}))_{\vec{F}} \to (\mathscr{X}, H_*(\mathbf{X})),
\end{array}$$

which is cartesian since it takes the canonical form of a cartesian morphism. □

We can also give the following characterization of cartesian functors between categories of hybrid objects.

Lemma 2.5. *If* $F : \mathsf{Hy}(\mathsf{C}) \to \mathsf{Hy}(\mathsf{D})$ *is a cartesian functor, then it restricts to a functor*

$$F_{\mathscr{D}} : \mathsf{C}^{\mathscr{D}} \to \mathsf{D}^{\mathscr{D}}$$

for every D-category $\mathscr{D}$.

[1] Note that we use the symbol "Π" to denote the projection functor for both $\mathsf{Hy}(\mathsf{C})$ and $\mathsf{Hy}(\mathsf{D})$ since is is the "canonical" projection onto the first factor in both cases.

2.5.7 Cartesian adjunctions. We can use the alternative perspective afforded by viewing categories of hybrid objects as fibered categories to better understand adjunctions between categories of hybrid objects.

Theorem 2.8. *Let* $L : \mathsf{Hy}(\mathsf{C}) \to \mathsf{Hy}(\mathsf{D})$ *and* $R : \mathsf{Hy}(\mathsf{D}) \to \mathsf{Hy}(\mathsf{C})$ *be a pair of cartesian functors. If the restriction of these functors:*

$$L_{\mathscr{D}} : \mathsf{C}^{\mathscr{D}} \rightleftarrows \mathsf{D}^{\mathscr{D}} : R_{\mathscr{D}}$$

is an adjunction for every D-category $\mathscr{D}$*, then*

$$L : \mathsf{Hy}(\mathsf{C}) \rightleftarrows \mathsf{Hy}(\mathsf{D}) : R$$

is an adjunction.

Proof. Analogous to the proof of Theorem 2.7, except now one constructs the natural bijection for the adjunction $L : \mathsf{Hy}(\mathsf{C}) \rightleftarrows \mathsf{Hy}(\mathsf{D}) : R$ by using the natural bijection for the fiberwise adjunctions. □

Chapter 3

Hybrid Model Structures

Model categories provide a method for doing "homotopy theory" on general categories. They were first introduced by Quillen [21] in order to axiomatize homotopy theory, i.e., a model category is a category with three types of distinguished morphisms—weak equivalences, fibrations and cofibrations–that satisfy certain axioms. Since his seminal paper in 1967, model category theory has blossomed into a full-fledged area of research capable of addressing homotopy-theoretic questions in a general context. Some of the quintessential model categories are the category of topological spaces, the category of simplicial sets and the category of chain complexes—the model structure of these categories plays a fundamental role in algebraic topology and homology.

We are interested in exploring the theory of model categories in the light of hybrid objects. This amounts to, for a D-category $\mathscr{D}$, finding a *homotopy meaningful* model structure on $\mathsf{M}^{\mathscr{D}}$ given a model structure on M. The homotopy theories for $\mathsf{M}^{\mathscr{D}}$ and M can then be related via homotopy colimits, i.e., one can relate "hybrid homotopy theory" to "non-hybrid homotopy theory" through the use of homotopy colimits. Therefore, the problem that must be addressed in order to understand hybrid model structures was first raised, in a more general form, by Grothendieck:

> *If* M *is a model category and* J *a small category, find a homotopy meaningful model structure(s) on* M^{J}.

The word "homotopy meaningful" in this statement needs some explanation, as it forms a central concept. In particular, there are two distinct homotopy meaningful model structures in which we are particularly interested, termed *cofibrantly homotopy meaningful* and *fibrantly homotopy meaningful*; these are related to the construction of homotopy colimits and homotopy limits, respectively.

In the first case, the goal is a model category structure that yields homotopy colimits—the total left derived functor of colim. That is, we want a model category structure on M^{J} in which:

> *For every weak equivalence* $f : X \xrightarrow{\sim} Y$ *between cofibrant objects* X *and* Y *in* M^{J}, $\mathrm{colim}_{\mathsf{J}}(f)$ *is a weak equivalence.*

The model structure on M^{J} is thus said to be *cofibrantly homotopy meaningful.* For such a model structure, the colimit induces a functor

$$\text{hocolim}_{\mathsf{J}} : \mathsf{Ho}(\mathsf{M}^{\mathsf{J}}) \to \mathsf{Ho}(\mathsf{M})$$

between homotopy categories, termed the *homotopy colimit*, which is given by

$$\text{hocolim}_{\mathsf{J}}(X) \cong \text{colim}_{\mathsf{J}}(X')$$

with X' any cofibrant object weakly equivalent to X.

In the second case, the goal is a model category structure that yields homotopy limits—the total right derived functor of lim. That is, we want a model category structure on M^{J} in which:

> *For every weak equivalence $f : X \xrightarrow{\bullet} Y$ between fibrant objects X and Y in M^{J}, $\lim_{\mathsf{J}}(f)$ is a weak equivalence.*

The model structure on M^{J} is thus said to be *fibrantly homotopy meaningful.* For such a model structure, the limit induces a functor:

$$\text{holim}_{\mathsf{J}} : \mathsf{Ho}(\mathsf{M}^{\mathsf{J}}) \to \mathsf{Ho}(\mathsf{M})$$

termed the *homotopy limit* and given by

$$\text{holim}_{\mathsf{J}}(X) \cong \lim_{\mathsf{J}}(X')$$

with X' any fibrant object weakly equivalent to X.

The problem posed by Grothendieck has received a lot of attention. In its full generality, and according to Dwyer [10]:

> *If J is an arbitrary small category, "it seems unlikely that M^{J} has a natural model category structure for a general model category M."*

This has spawned a variety of approaches to understanding the model structure of M^{J}—the three most prominent being:

- *Considering small categories J with special shapes.* This is the approach taken in the expository paper by Dwyer [10] where he puts a model category structure on very small categories J; this structure has appeared in papers relating to dérivateurs(cf. [12] and [13]). A similar approach is taken in [8], [16] and [17], where Reedy categories are considered. Similarly, the work by [7] considers small categories J obtained from simplicial sets; the authors are then able to define homotopy colimits and homotopy Kan extensions through the use of model approximations.
- *Considering model categories M with special structures.* This is, for example, the approach taken by [14] where the author considers cosimplicial model categories.
- *Considering modified notions of model categories.* This approach modifies the notion of a model category so that it is possible to extend this structure to functor categories. In [8] the notion of a homotopical category is defined, and it is shown how to construct homotopy colimits. Similarly, in [27], Thomason model categories are introduced with the goal of defining homotopy colimits.

Fortunately, in the case when J is a D-category, the partial answers to Grothendieck's question are sufficient for our purposes. Given a model category M, there is a cofibrantly homotopy meaningful model structure on $\mathsf{M}^{\mathscr{D}}$. Therefore, the homotopy colimit:

$$\text{hocolim}_{\mathscr{D}} : \mathsf{Ho}(\mathsf{M}^{\mathscr{D}}) \to \mathsf{Ho}(\mathsf{M})$$

exists, and thus relates "hybrid homotopy theory" to "non-hybrid homotopy theory." The connection between hybrid objects and hybrid systems allows for the formulation of a *homotopy theory of hybrid systems* (with the same discrete structure). This is the main contribution of this thesis, applications of which are discussed in the context of topology and homology.

3.1 Model Categories

This section introduces the basics of model categories, including the axioms defining a model category together with the homotopy category associated to a model category. Examples will follow in the next section.

3.1.a Basics

We begin by reviewing the definition of a model category; the one introduced here follows from [10]. Note that this definition of a model category corresponds to the notion of a closed model category as introduced by Quillen [21], except that we strengthen **MC1** by assuming that M is complete and cocomplete, i.e., small limits and colimits exist in M.

3.1.1 Model categories. A model category M is a category with three special classes of morphisms:

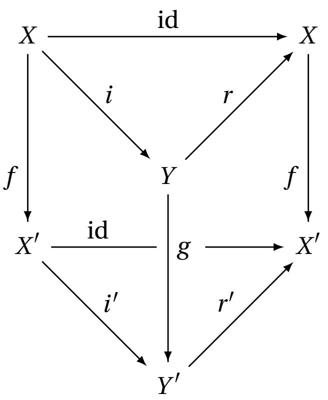
- weak equivalences (denoted by $\xrightarrow{\sim}$),
- fibrations (denoted by $\twoheadrightarrow$),
- cofibrations (denoted by $\rightarrowtail$),

which are closed under composition and contain all identity morphisms. In addition, the following five axioms must be satisfied:

MC1: M is complete and cocomplete.

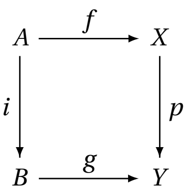
MC2: For two morphisms f and g in M such that $g \circ f$ is defined, if any two of the three morphisms $f, g, g \circ f$ are weak equivalences then so is the third.

MC3: For every commuting diagram of the form:



in which g is a weak equivalence, fibration or cofibration, f is a weak equivlance, fibration or cofibration, respectivly.

MC4: For every commutative diagram of the form:



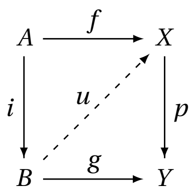
(3.1)

if either of the following conditions hold:

(1) i is a cofibration and p is an acyclic fibration,

(2) i is an acyclic cofibration and p is a fibration.

then there exists a $u : B \to X$ such that the following diagram



commutes.

MC5: Any morphism f can be factored in the following two ways:

(1) $f = p \circ i$ where i is a cofibration and p is an acyclic fibration,

(2) $f = p \circ i$ where i is an acyclic cofibration and p is a fibration.

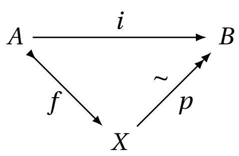
To provide evidence of the tight coupling between the different axioms in the definition a model category, we introduce some fundamental statements related to the characterization and stability properties of fibrations and cofibrations.

3.1.2 Lifting properties. For the commutative square given in (3.1) we say that:

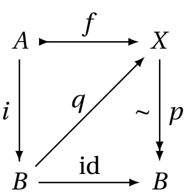
LLP: $i : A \to B$ has the left lifting property (LLP) with respect to p if a lift $u : B \to X$ exists.
RLP: $p : X \to Y$ has the right lifting property (RLP) with respect to i if a lift $u : B \to X$ exists.

Lifting properties can be used to characterize cofibrations and fibrations.

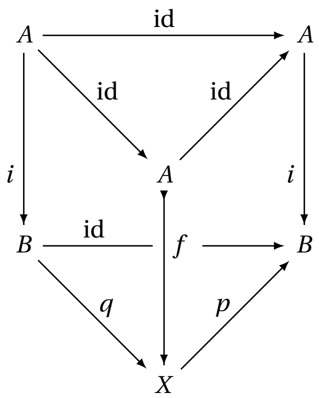
To provide a specific example of this, let $i : A \to B$ be a morphism with the LLP with respect to all acyclic fibrations. This morphism factors as:



Which yields a commutative diagram



The end result is a retract diagram:



It follows from **MC3** that $i : A \to B$ is a cofibration. Therefore, i is a cofibration iff it has the LLP with respect to acyclic fibrations. Similar arguments show (cf. [10]) that:

Proposition 3.1. *If* M *be a model category, then a morphism* $i : A \to B$ *is*

(i) *A cofibration iff it has the LLP with respect to all acyclic fibrations.*

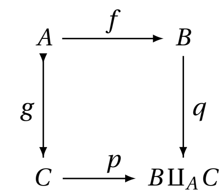
(ii) *An acyclic cofibration iff it has the LLP with respect to all fibrations.*

Dually, a morphism $p : X \to Y$ *is*

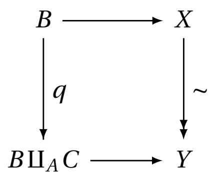
(i) *A fibration iff it has the RLP with respect to all acyclic cofibrations.*

(ii) *An acyclic fibration iff it has the RLP with respect to all cofibrations.*

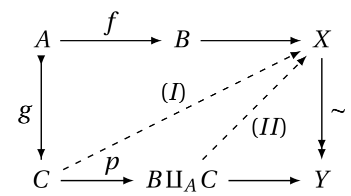
3.1.3 Stability properties. Cofibrations are stable under cobase change and fibrations are stable under base change. Specifically, consider the pushout square:



where g is a cofibration. The claim is that q is a cofibration, i.e., for every diagram of the form:



a lift exists (see Proposition 3.1). To see this, note that the two above diagrams yield a commuting diagram:



where this existence of (II) follows from the existence of the (I) by the universality of the pushout. Through similar arguments, one concludes that:

Lemma 3.1. *For the pushout square:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{p} & B \amalg_A C \end{array}$$

(i) *If f is a (acyclic) cofibration, then p is a (acyclic) cofibration.*

(ii) *If g is a (acyclic) cofibration, then q is a (acyclic) cofibration.*

The dual of this statement is:

Lemma 3.2. *For the pullback square:*

$$\begin{array}{ccc} B \times_A C & \xrightarrow{j} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & A \end{array}$$

(i) *If f is a (acyclic) fibration, then i is a (acyclic) fibration.*

(ii) *If g is a (acyclic) fibration, then j is a (acyclic) fibration.*

3.1.b The Homotopy Category

With the concept of a model category in hand, we introduce the associated homotopy category. In order to do so, the fundamental concept of cofibrant and fibrant replacements must be introduced.

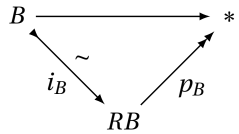
3.1.4 Cofibrant and fibrant objects. An object A of M is said to be *cofibrant* if the morphism from the initial object of M to A is a cofibration: $\emptyset \rightarrowtail A$. An object B of M is *fibrant* if the morphism from B to the terminal object is a fibration: $B \twoheadrightarrow *$.

Given an object A of M, we can use **MC5** to define its *cofibrant replacement.* That is, for the morphism $\emptyset \to A$, there is a factorization:

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ & {\scriptstyle i_A}\searrow \quad \nearrow{\scriptstyle p_A}\,\sim & \\ & QA & \end{array}$$

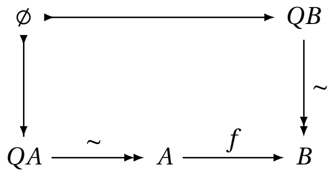
Therefore, QA is a cofibrant object weakly equivalent to A, termed the cofibrant replacement of A.

Similarly, for an object B of M and the morphism $B \to *$, there is a factorization:



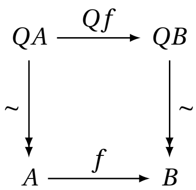
Therefore, RB is a fibrant object weakly equivalent to B, termed the fibrant replacement of B.

Fibrant and cofibrant replacements are functorial in the following sense: let $f : A \to B$, then there is a commuting diagram:



It follows from **MC4** and **MC2** that:

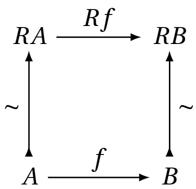
Lemma 3.3. *For $f : A \to B$ there exists a morphism $Qf : QA \to QB$ making the following diagram commute:*



and f is a weak equivalence iff Qf is a weak equivalence.

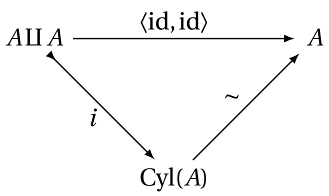
The dual of this statement also can be given:

Lemma 3.4. *For $f : A \to B$ there exists a morphism $Rf : RA \to RB$ making the following diagram commute:*

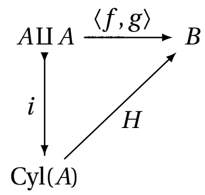


and f is a weak equivalence iff Rf is a weak equivalence.

3.1.5 Left homotopies. Consider the morphism $\langle \mathrm{id}, \mathrm{id} \rangle : A \amalg A \to A$. A *cylinder object*, denoted by $\mathrm{Cyl}(A)$, is any object such that there is a factorization:

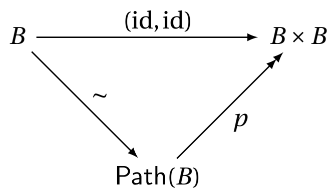


Cylinder objects exist by **MC5**. A left homotopy from $f : A \to B$ to $g : A \to B$ is a morphism $H : \mathrm{Cyl}(A) \to B$ making the following diagram

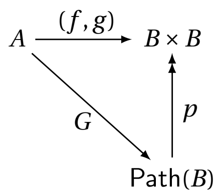


commute. In this case, we write $f \overset{l}{\simeq} g$.

3.1.6 Right homotopies. Consider the morphism $(\mathrm{id},\mathrm{id}) : B \to B \times B$. A *path object*, denoted by $\mathsf{Path}(B)$, is any object such that there is a factorization:



A right homotopy from $f : A \to B$ to $g : A \to B$ is a morphism $G : A \to \mathsf{Path}(B)$ making the following diagram commute:



In this case, we write $f \overset{r}{\simeq} g$.

The notions of left and right homotopies agree in certain cases (Lemma 4.21, [10]):

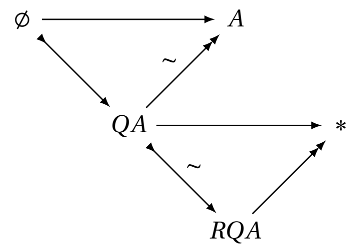
Lemma 3.5. *If $f, g : A \to B$, with A cofibrant and B fibrant, then $f \overset{r}{\simeq} g$ iff $f \overset{l}{\simeq} g$.*

3.1.7 The homotopy category. If $f, g : A \to B$, with A cofibrant and B fibrant, then because of the above proposition we write $f \simeq g$, and say that f is homotopic to g. This forms an equivalence relation on $\mathsf{Hom}_{\mathsf{M}}(A, B)$; therefore, define

$$\pi(A,B) = \mathsf{Hom}_{\mathsf{M}}(A,B)/\sim$$

where $[f] \in \pi(A,B)$ is given by $[f] = \{g \in \mathsf{Hom}_{\mathsf{M}}(A,B) : f \simeq g\}$.

Before defining the homotopy category, note that the fibrant-cofibrant replacement RQA of an object of A is simultaneously fibrant and cofibrant. To see this, note that there is a commuting diagram:

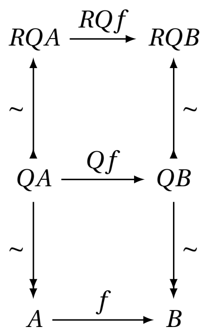


So, in fact, RQA is weakly equivalent to A. We now define:

Definition 3.1. The *homotopy category* $\mathsf{Ho}(\mathsf{M})$ of a model category M is a category with the same objects as M and with

$$\mathsf{Hom}_{\mathsf{Ho}(\mathsf{M})}(A, B) = \pi(RQA, RQB).$$

Recall that for $f : A \to B$ (A and B arbitrary objects of M), we have a commuting diagram:



Therefore, we can define a functor

$$\gamma : \mathsf{M} \to \mathsf{Ho}(\mathsf{M})$$

with $\gamma(A) = A$ for all objects of A of M and $\gamma(f) = [RQf]$. (See [21] for more on the structure of homotopy categories.)

The following lemma relates homotopies with weak equivalences:

Lemma 3.6. *If $f : A \to B$ with A and B both fibrant and cofibrant, then f is a weak equivalence iff f is homotopic to the identity, i.e., there exists a morphism $g : B \to A$ such that $f \circ g \simeq \mathrm{id}_B$ and $g \circ f \simeq \mathrm{id}_A$.*

From which, we get the related result:

Lemma 3.7. *$f : A \to B$ is a weak equivalence in M iff $\gamma(f) : A \to B$ is an isomorphism in $\mathsf{Ho}(\mathsf{M})$.*

3.2 Examples

This section introduces a few of the fundamental examples of model categories: simplicial sets, topological spaces and chain complexes; all of these examples of model categories have been well-studied, even back to Quillen's original work [21] on model categories. While it is possible to state in a concise fashion the model structure of these categories, proving that the axioms hold is no trivial matter. In the case of topological spaces and chain complexes, we refer the reader to [10]. Bousfield and Kan [5] provide a nice summary of the model structure on simplicial sets.

3.2.a Topological Spaces

We begin by considering the category of topological spaces and (one of) its model structures; there are other model structures, but it is fair to say that the one introduced here is the "standard" one. Our treatment of the model structure of topological spaces follows closest to [10] and [17].

3.2.1 A model structure on Top**.** Let Top be the category of topological spaces. Let X and Y be topological spaces with basepoints x and y, i.e., X and Y are pointed topological spaces. A map of pointed topological spaces is a continuous function $f : X \to Y$ such that $f(x) = y$. Two maps $f, g : X \to Y$ of pointed topological spaces are homotopic if there exists a map $H : X \times I \to Y$ such that $H(x', 0) = f(x')$, $H(x', 1) = g(x')$ for all $x' \in X$ and $H(x, t) = y$ for all $t \in I$ (with x the basepoint of X and y the basepoint of Y). This defines an equivalence relation.

For a topological space X, pick a basepoint $x \in X$. The n^{th} homotopy set of X at x, $\pi_n(X, x)$ is the set of homotopy classes of pointed maps:

$$(\mathbb{S}^n, * = (1, 0, \ldots, 0)) \to (X, x).$$

This is a group for $n \geq 1$. A map $f : X \to Y$ of topological spaces is said to be a *weak homotopy equivalence* if for all $x \in X$ the induced map:

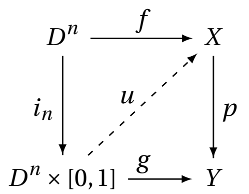
$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x)),$$

is a bijection of sets for $n = 0$ and an isomorphism of groups for $n \geq 1$.

A map $p : X \to Y$ is a *Serre fibration* if it has the RLP (3.1.2) with respect to the maps:

$$\begin{aligned} i_n : D^n &\to D^n \times [0,1] \\ x &\mapsto (x, 0), \end{aligned}$$

for $n \geq 0$; here D^n is the standard n-disk and $D^0 = \{0\}$, i.e., for every commuting diagram:



the map u exists and is unique for all $n \geq 0$.

With these definitions in hand, it has been shown (cf. [10]) that there is the following model structure on Top.

Theorem 3.1. *The category of topological spaces,* Top*, is a model category with the following choices of weak equivalences, fibrations and cofibrations:* $f : X \to Y$ *is*

We. *A weak equivalence if it is a weak homotopy equivalence,*

Fib. *A fibration if it is a Serre fibration,*

Cof. *A cofibration if it has the LLP with respect to acyclic fibrations.*

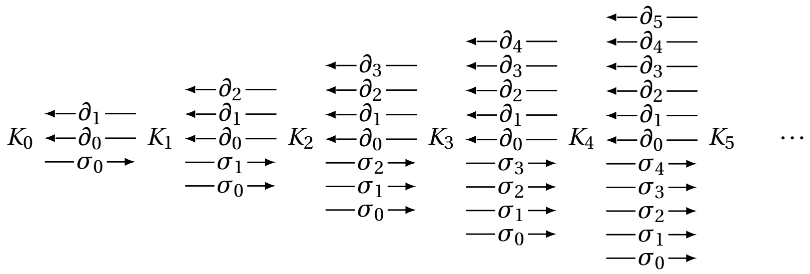
3.2.b Simplicial Sets

We now turn our attention toward simplicial sets; it is sometimes easier to work with this category as it essentially consists of "combinatorial" data. Moreover, and although Top and SSet are very different categories, it will be seen that their "homotopy theories" are equivalent. For more on simplicial sets, we refer the reader to the excellent references [5] and [19] on the subject.

3.2.2 Simplicial objects. Let C be a category. A simplicial object K over C consists of a sequence of objects K_n, $n \geq 0$, together with face morphisms, $\partial_i : K_n \to K_{n-1}$, and degeneracy morphisms, $\sigma_i : K_n \to K_{n+1}$, for $i = 0, 1, \ldots, n$, such that these morphisms satisfy the classical *simplicial identities*:

$$
\begin{array}{rcl}
\partial_i \partial_j & = & \partial_{j-1} \partial_i \quad \text{if} \quad i < j \\
\sigma_i \sigma_j & = & \sigma_j \sigma_{i-1} \quad \text{if} \quad i > j \\
\partial_i \sigma_j & = & \begin{cases} \sigma_{j-1} \partial_i & \text{if} \quad i < j \\ \text{id} & \text{if} \quad i = j \text{ or } i = j+1 \\ \sigma_j \partial_{i-1} & \text{if} \quad i > j+1. \end{cases}
\end{array} \tag{3.2}
$$

Simplicial objects can be visualized as follows:



A morphism of simplicial objects K and L, $f : K \to L$, is just a collection of morphisms $f_n : K_n \to$

L_n $(n \geq 0)$ in C such that the following diagrams

$$\begin{array}{ccc} K_{n-1} & \xleftarrow{\partial_i^K} & K_n \\ \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} \\ L_{n-1} & \xleftarrow{\partial_i^L} & L_n \end{array} \qquad\qquad \begin{array}{ccc} K_n & \xrightarrow{\sigma_i^K} & K_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ L_n & \xrightarrow{\sigma_i^L} & L_{n+1} \end{array}$$

commute.

Specific examples of simplicial objects that are of interest are simplicial sets, SSet, and simplicial abelian groups, SAb. In the case of a simplicial set K, an element of K_n is termed an n-simplex.

3.2.3 Geometric realization. Recall (see [5], [19] and [20]) that taking the geometric realization of a simplicial set results in a functor

$$|-|: \mathsf{SSet} \to \mathsf{Top}$$

In particular, a morphism $f: K \to L$ of simplicial sets induces a morphism $|f|: |K| \to |L|$ of topological spaces.

There is the following model structure on SSet established by [5] (see also [10] and [21]).

Theorem 3.2. *The category of simplicial sets,* SSet*, is a model category with the following choices of weak equivalences, fibrations and cofibrations: $f: K \to L$ is a*

We. *A weak equivalence if $|f|: |K| \to |L|$ is a weak homotopy equivalence,*

Fib. *A fibration if f has the RLP with respect to acyclic cofibrations,*

Cof. *A cofibration if each $f_n: K_n \to L_n$ $(n \geq 0)$ is injective.*

3.2.4 Singular functor. To every topological space, there is an associated simplicial set; this defines a functor

$$\mathrm{sing}: \mathsf{Top} \to \mathsf{SSet}.$$

The singular and realization functor relate the model structure on Top and SSet as follows (see [5]):

Proposition 3.2. *For the model structure on* Top *given in Theorem 3.1 and the model structure on* SSet *given in Theorem 3.2, the following conditions hold:*

- *The functors $|-|$ and* sing *preserve weak equivalences.*
- *The functor $|-|$ preserves fibrations and cofibrations.*

3.2.5 Simplicial sets from small categories. Given a small category J, we can associate to this small category a simplicial set called the *nerve* of J and denoted by $N\mathsf{J}$ (this exposition follows [11], but this construction is widely known and also can be found in [5],[14],[22],[24]-[25]). In particular, define the *nerve* functor as a functor

$$N\colon \mathsf{Cat} \to \mathsf{SSet},$$

where for every sequence of n composable morphisms

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} a_{n-1} \xrightarrow{\alpha_n} a_n$$

in J, we define the n-simplex $\alpha = (a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n)$ in $N\mathsf{J}_n$; $N\mathsf{J}_n$ is the set of all n-simplicies of this form. The 0-simplicies in $N\mathsf{J}$, i.e., the elements of $N\mathsf{J}_0$, are the objects of J. The face morphisms are defined by

$$\begin{aligned}
\alpha \in N\mathsf{J}_1 \qquad & \partial_0\alpha = \partial_0(a_0 \xrightarrow{\alpha_1} a_1) = a_1\\
& \partial_1\alpha = \partial_1(a_0 \xrightarrow{\alpha_1} a_1) = a_0
\end{aligned}$$

$$\begin{aligned}
\alpha \in N\mathsf{J}_n \qquad & \partial_0\alpha = \partial_0(a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n) = (a_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} a_n)\\
0 < i < n \qquad & \partial_i\alpha = \partial_i(a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n)\\
& \quad = (a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} a_{i-1} \xrightarrow{\alpha_{i+1}\circ\alpha_i} a_{i+1} \xrightarrow{\alpha_{i+2}} \cdots \xrightarrow{\alpha_n} a_n)\\
& \partial_n\alpha = \partial_0(a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} a_{n-1} \xrightarrow{\alpha_n} a_n) = (a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} a_{n-1})
\end{aligned}$$

and the degeneracy morphisms are defined by

$$\sigma_i(a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n) = (a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} a_i \xrightarrow{\mathrm{id}} a_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_n} a_n).$$

For simplicity when refereing to n-simplicies in $N\mathsf{J}$, for $\alpha \in N\mathsf{J}_n$, let

$$\mathsf{dom}(\alpha) = \mathsf{dom}(a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n) = a_0.$$

A functor $F\colon \mathsf{J} \to \mathsf{J}'$ induces a morphism $NF\colon N\mathsf{J} \to N\mathsf{J}'$ in the obvious manner, i.e.,

$$NF_n(\alpha) = NF_n(a_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} a_n) = F(a_0) \xrightarrow{F(\alpha_1)} \cdots \xrightarrow{F(\alpha_n)} F(a_n).$$

The importance of this construction is that it allows us to apply techniques utilized in the study of simplicial sets to small categories. For example, one can consider the geometric realization of the nerve of a small category, $|N\mathsf{J}|$. This yields a topological space termed the *classifying space* of J.

3.2.c Chain Complexes

The third, and final, example of a model category is the category of (non-negative) chain complexes of abelian groups. Note that we could instead consider general abelian categories, but this would result in an unnecessary level of abstraction for the constructions that will be considered.

3.2.6 Chain complexes. Let Ab be the category of abelian groups. The category of non-negative chain complexes $\mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ has as objects chain complexes $A_\bullet$ consisting of a family of abelian groups, $\{A_n\}_{n\geq 0}$, together with morphisms $d_n : A_n \to A_{n-1}$ (typically denoted by d, with the index understood) such that $d^2 = d \circ d = 0$; here $d_0 = 0$. Therefore, $A_\bullet$ can be visualized as follows:

$$A_\bullet : \qquad \cdots \xleftarrow{d} A_{n-1} \xleftarrow{d} A_n \xleftarrow{d} A_{n+1} \xleftarrow{d} \cdots$$

A morphisms of chain complexes, $f : A_\bullet \to B_\bullet$ is a collection of morphisms (in Ab) $f_n : A_n \to B_n$ for $n \geq 0$ such that the following diagram:

$$\begin{array}{lccccccccc} A_\bullet : & \cdots & \xleftarrow{d} & A_{n-1} & \xleftarrow{d} & A_n & \xleftarrow{d} & A_{n+1} & \xleftarrow{d} & \cdots \\ & & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} & & \\ B_\bullet : & \cdots & \xleftarrow{d} & B_{n-1} & \xleftarrow{d} & B_n & \xleftarrow{d} & B_{n+1} & \xleftarrow{d} & \cdots \end{array}$$

commutes.

For a chain complex $A_\bullet$, its n^{th} homology group is given by:

$$H_n(A_\bullet) = \frac{\mathrm{Ker}(d_n)}{\mathrm{Im}(d_{n+1})}.$$

This is well-defined because $d \circ d = 0$. Also note that a morphism of chain complexes $f : A_\bullet \to B_\bullet$ induces a morphism of homology groups: $H_n(f) : H_n(A_\bullet) \to H_n(B_\bullet)$. This implies that H_n is a functor:

$$H_n : \mathsf{Ch}_{\geq 0}(\mathsf{Ab}) \to \mathsf{Ab}.$$

for all $n \geq 0$.

With these definitions, there is the following model category.

Theorem 3.3. *The category of non-negative chain complexes of abelian groups,* $\mathsf{Ch}_{\geq 0}(\mathsf{Ab})$*, is a model category with the following choices of weak equivalences, fibrations and cofibrations:* $f : A_\bullet \to B_\bullet$ *is*

We. *A weak equivalence if for all* $n \geq 0$, $H_n(f) : H_n(A_\bullet) \to H_n(B_\bullet)$ *is an isomorphism,*

Fib. *A fibration if for each* $n \geq 1$, $f_n : A_n \to B_n$ *is an epimorphism,*

Cof. *A cofibration if for each* $n \geq 0$, $f_n : K_n \to L_n$ *is a monomorphism with a free abelian group as its cokernal.*

The model structure on $\mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ was first introduced in [21]; see also [10]. In these references, chain complexes over abelian categories and R-models were considered, respectively. In these cases, cofibrations are monomorphisms with projective objects as cokernals. Part of the motivation for considering abelian groups is that in Ab an object is projective iff it is free.

3.2.7 The homology of simplicial abelian groups. Consider the category SAb of simplicial abelian groups. For every simplicial abelian group A there is an associated complex called the *Moore complex* which will be denoted by $A_\bullet$. In particular, we have

$$A_\bullet : \qquad \cdots \xleftarrow{d} A_{n-1} \xleftarrow{d} A_n \xleftarrow{d} A_{n+1} \xleftarrow{d} \cdots$$

where

$$d = \sum_{i=0}^{n} (-1)^i \partial_i : A_n \to A_{n-1}.$$

From the simplicial identities it follows that $d^2 = 0$. The homology of A is thus given by $H_n(A) := H_n(A_\bullet)$.

The Moore complex allows one to consider the homology of a simplicial set. Let $\mathbb{Z}\langle - \rangle : \mathsf{Set} \to \mathsf{Ab}$ be the functor that associates to a set the free abelian group generated by the elements of the set. This induces a functor $\mathbb{Z}\langle - \rangle : \mathsf{SSet} \to \mathsf{SAb}$, denoted by the same symbol. Therefore, the homology of a simplicial set, K, is defined to be

$$H_n(A) := H_n(\mathbb{Z}\langle K\rangle_\bullet),$$

where $\mathbb{Z}\langle K\rangle_\bullet$ is the chain complex associated to the simplicial abelian group $\mathbb{Z}\langle K\rangle$.

Through these constructions, one can define the homology of a topological space. Given a topological space X, its (singular) homology is defined to be

$$H_n(X) := H_n(\mathbb{Z}\langle \mathrm{sing}(X)\rangle_\bullet).$$

This defines a functor $H_n : \mathsf{Top} \to \mathsf{Ab}$.

In a similar manner, composing the functors:

$$\mathsf{Cat} \xrightarrow{N} \mathsf{SSet} \xrightarrow{\mathbb{Z}\langle - \rangle} \mathsf{SAb}$$

allows one to define the homology of a small category:

$$H_n(\mathsf{J}) := H_n(\mathbb{Z}\langle N\mathsf{J}\rangle_\bullet).$$

In fact, since the morphism $N\mathsf{J} \to \mathrm{sing}(|N\mathsf{J}|)$ is a weak equivalence (see Example 3.1), and because weak equivalences in the category of topological spaces induce isomorphisms on homology,

$$H_n(\mathsf{J}) \cong H_n(|N\mathsf{J}|).$$

That is, one obtains the homology of a small category by composing the functors:

$$\mathsf{Cat} \xrightarrow{N} \mathsf{SSet} \xrightarrow{|\,-\,|} \mathsf{Top} \xrightarrow{H_n} \mathsf{Ab}$$

This nicely demonstrates the interplay between the different concepts involved.

The following construction is related to associating to a simplicial abelian group its Moore complex.

3.2.8 The homology of a small category with coefficients in a functor. Given a functor $L\colon \mathsf{J} \to \mathsf{Ab}$, a simplicial abelian group can be constructed; denote this simplicial abelian group by J^L and define

$$\mathsf{J}^L_n = \bigoplus_{\alpha \in N\mathsf{J}_n} L(\mathsf{dom}(\alpha)).$$

The face morphisms, ∂_i, and the degeneracy morphisms, σ_i, are given by requiring that the following diagrams

$$\begin{array}{ccc}
\bigoplus_{\alpha\in N\mathsf{J}_{n-1}} L(\mathsf{dom}(\alpha)) & \xleftarrow{\ \partial_i\ } & \bigoplus_{\alpha\in N\mathsf{J}_n} L(\mathsf{dom}(\alpha)) \\
\uparrow{\scriptstyle \iota_{\partial_i^{N\mathsf{J}}\alpha}} & & \uparrow{\scriptstyle \iota_\alpha} \\
L(\mathsf{dom}(\partial_i^{N\mathsf{J}}\alpha)) & \xleftarrow{\left\{\begin{array}{lll} L(\alpha_1) & \text{if} & i=0 \\ \text{id} & \text{if} & 0<i\le n \end{array}\right\}} & L(\mathsf{dom}(\alpha))
\end{array}$$

$$\begin{array}{ccc}
\bigoplus_{\alpha\in N\mathsf{J}_n} L(\mathsf{dom}(\alpha)) & \xrightarrow{\ \sigma_i\ } & \bigoplus_{\alpha\in N\mathsf{J}_{n+1}} L(\mathsf{s}(\alpha)) \\
\uparrow{\scriptstyle \iota_\alpha} & & \uparrow{\scriptstyle \iota_{\sigma_i^{N\mathsf{J}}\alpha}} \\
L(\mathsf{dom}(\alpha)) & \xrightarrow{\ \text{id}\ } & L(\mathsf{dom}(\sigma_i^{N\mathsf{J}}\alpha))
\end{array}$$

commute; here $\partial_i^{N\mathsf{J}}$ and $\sigma_i^{N\mathsf{J}}$ are the face and degeneracy morphisms of $N\mathsf{J}$ and ι_α are the inclusion morphisms into the direct sum. This allows us to define the homology of a small category with coefficients in a functor.

Definition 3.2. For a functor $L\colon \mathsf{J} \to \mathsf{Ab}$,

$$H_n(\mathsf{J}, L) := H_n(\mathsf{J}^L_\bullet).$$

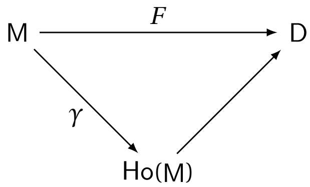
Given the relationships established in Paragraph 3.2.7, one can define the homology of a small category with coefficients in more general diagrams. For example, if $L\colon \mathsf{J} \to \mathsf{Top}$, then we can consider the homology group $H_n(\mathsf{J}, H_p(L))$, with $H_p(L)$ the composite of L and H_p. This will be discussed in further detail in Paragraph 3.3.8.

3.3 Quillen Adjunctions

We now introduce the notion of a Quillen adjunction which is fundamental in understanding the interplay between different model categories. This follows from the fact that adjunctions of this form imply the existence of total (left and right) derived functors and thus induce an adjunction between homotopy categories. These concepts are solidified by considering the categories Top and SSet.

3.3.a Derived Functors

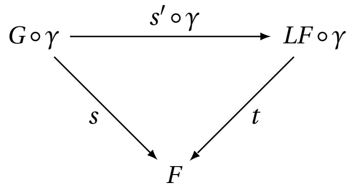
Consider a functor $F : \mathsf{M} \to \mathsf{D}$ with M a model category. In general, this functor does not factor through the homotopy category of M, i.e., there does *not* exist a factorization:



Left and right derived functors are introduced in order to find the "closest approximation" to such a factorization "from the left" or "from the right."

Following from [10] and [21], we define:

3.3.1 Left derived functors. Let $F : \mathsf{M} \to \mathsf{D}$ be a functor, with M a model category. A left derived functor of F is a pair (LF, t) where $LF : \mathrm{Ho}(\mathsf{M}) \to \mathsf{D}$ and $t : LF \circ \gamma \xrightarrow{\cdot} F$ where, again, $\gamma : \mathsf{M} \to \mathrm{Ho}(\mathsf{M})$. In addition, it must satisfy the universal property that for any $G : \mathrm{Ho}(\mathsf{M}) \to \mathsf{D}$ and any $s : G \circ \gamma \xrightarrow{\cdot} F$ there exists a unique $s' : G \xrightarrow{\cdot} LF$ such that the following diagram



commutes. The following result (cf. [21]) is very useful:

Proposition 3.3. *Let $F : \mathsf{M} \to \mathsf{D}$ with M a model category. If $F(f)$ is an isomorphism whenever f is a weak equivalence between cofibrant objects, then the left derived functor (LF, t) of F exists and for every cofibrant object A of M, the morphism $t_A : LF(A) \to F(A)$ is an isomorphism.*

3.3.2 Total left derived functors. A total left derived functor of a functor $F : \mathsf{M} \to \mathsf{N}$ between model categories is a functor

$$\mathbb{L}F : \mathrm{Ho}(\mathsf{M}) \to \mathrm{Ho}(\mathsf{N})$$

such that $\mathbb{L}F$ is a left derived functor of the composite:

$$\begin{array}{ccc} \mathsf{M} & \xrightarrow{\gamma_{\mathsf{N}} \circ F} & \mathsf{Ho}(\mathsf{N}) \\ & {\scriptstyle F}\searrow \quad \nearrow{\scriptstyle \gamma_{\mathsf{N}}} & \\ & N & \end{array}$$

We have the following very important corollary to Proposition 3.3.

Corollary 3.1. *If $F : \mathsf{M} \to \mathsf{N}$ preserves weak equivalences between cofibrant objects, then the total left derived functor $\mathbb{L}F : \mathsf{Ho}(\mathsf{M}) \to \mathsf{Ho}(\mathsf{N})$ exists and can be computed by:*

$$\mathbb{L}F(A) \cong F(A')$$

for any cofibrant object A' weakly equivalent to A.

Right derived functors and total right derived functors are defined dually. If a functor $F : \mathsf{M} \to \mathsf{N}$ between model categories preserves weak equivalences between fibrant objects, then the total right derived functor $\mathbb{R}F : \mathsf{Ho}(\mathsf{M}) \to \mathsf{Ho}(\mathsf{N})$ exists and can be computed by:

$$\mathbb{R}F(A) \cong F(A')$$

for any fibrant object A' weakly equivalent to A.

3.3.b Quillen Adjunctions

Quillen adjunctions are related to the existence of total left and right derived functors, and thus play a fundamental role in model category theory.

Definition 3.3. A *Quillen adjunction* is an adjunction between model categories:

$$F : \mathsf{M} \rightleftarrows \mathsf{N} : G \tag{3.3}$$

such that

(i) F preserves cofibrations and acyclic cofibrations,

(ii) G preserves fibrations and acyclic fibrations.

To better understand when an adjunction $F : \mathsf{M} \rightleftarrows \mathsf{N} : G$ is a Quillen adjunction, suppose that F preserves acyclic cofibrations. Consider a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & G(X) \\ {\scriptstyle \sim}\big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle G(p)} \\ B & \longrightarrow & G(Y) \end{array}$$

with $p: X \to Y$ a fibration. Because F and G are adjoint functors, we have a commutative diagram

$$\begin{array}{ccc} F(A) & \longrightarrow & X \\ \sim \Big\downarrow F(i) & & \Big\downarrow p \\ F(B) & \longrightarrow & Y \end{array}$$

and so there exists a lift $F(B) \to X$. This implies the existence of a lift $B \to G(X)$ and so $G(p)$ is a fibration.

Therefore, this and a similar argument imply that:

Lemma 3.8. *If $F : \mathsf{M} \rightleftarrows \mathsf{N} : G$ is an adjunction, then the following statements are equivalent:*

(i) *F preserves cofibrations and acyclic cofibrations,*

(ii) *G preserves fibrations and acyclic fibrations.*

The definition of a Quillen adjunction is motivated by the following proposition:

Proposition 3.4. *Let M and N be model categories, and*

$$F : \mathsf{M} \rightleftarrows \mathsf{N} : G$$

be a pair of adjoint functors. Then if either of the following conditions hold:

(i) *F preserves cofibrations and acyclic cofibrations,*

(ii) *G preserves fibrations and acyclic fibrations,*

then the total derived functors:

$$\mathbb{L}F : \mathsf{Ho}(\mathsf{M}) \rightleftarrows \mathsf{Ho}(\mathsf{N}) : \mathbb{R}G$$

exist and form an adjoint pair.

Related to the proposition (and the proof thereof, see [10]), is the following lemma due to K. Brown:

Lemma 3.9. *Let $F : \mathsf{M} \to \mathsf{N}$ be a functor between model categories. If F carries acyclic cofibrations to weak equivalences, then F preserves weak equivalences between cofibrant objects.*

Dually, if F carries acyclic fibrations to weak equivalences, then F preserves weak equivalences between fibrant objects.

From which we have the following:

Corollary 3.2. *If $F : \mathsf{M} \rightleftarrows \mathsf{N} : G$ is a Quillen adjunction, then*

(i) *F preserves weak equivalences between cofibrant objects,*

(ii) *G preserves weak equivalences between fibrant objects.*

3.3.3 Quillen equivalences. It follows from Proposition 3.4 and Corollary 3.2 that, for a Quillen adjunction (3.3), the total derived functors:

$$\mathbb{L}F : \mathsf{Ho}(\mathsf{M}) \rightleftarrows \mathsf{Ho}(\mathsf{N}) : \mathbb{R}G \tag{3.4}$$

exist and form an adjoint pair. A Quillen adjunction is said to be a *Quillen equivalence* if (3.4) is an adjoint equivalence of categories. That is, for $\mathrm{id}_{F(A)} : F(A) \to F(A)$ with A in M and $\mathrm{id}_{G(X)} : G(X) \to G(X)$ for X in N, the morphisms:

$$\mathrm{id}^{\sharp}_{F(A)} : A \to G(F(A)), \qquad \mathrm{id}^{\flat}_{G(X)} : F(G(X)) \to X,$$

are weak equivalences.

Example 3.1. In [19], J. P. May proved that there is an adjunction:

$$|\ -\ | : \mathsf{SSet} \rightleftarrows \mathsf{Top} : \mathrm{sing}.$$

That is, there is a natural bijection

$$\mathsf{Hom}_{\mathsf{Top}}(|K|, X) \cong \mathsf{Hom}_{\mathsf{SSet}}(K, \mathrm{sing}(X))$$

for every simplicial set K and topological space X. Proposition 3.2 implies that this adjunction is a Quillen adjunction. Moreover, the morphisms:

$$X \to \mathrm{sing}(|X|), \qquad |\mathrm{sing}(Y)| \to Y,$$

are weak equivalences for all $X \in \mathsf{Ob}(\mathsf{SSet})$ and $Y \in \mathsf{Ob}(\mathsf{Top})$. Therefore, the adjunction is a Quillen equivalence. This implies, to quote Bousfield and Kan, that there is an "equivalence between homotopy theories of the categories SSet and Top."

3.3.c Homotopy Meaningful Model Category Structures

Let M denote a model category and J a small category. The goal is to give conditions on the model structure of M^{J}, if such a structure exists, so that it is cofibrantly or fibrantly homotopy meaningful. This indicates that there are *two* sperate homotopy meaningful model category structures for M^{J} depending on whether one is interested in considering homotopy colimits or homotopy limits. Of course, one could consider other functors $F : \mathsf{M}^{\mathsf{J}} \to \mathsf{M}$, and so define a more general notion of a homotopy meaningful model structure on M^{J}. This added generality does not seem to yield much additional benefit; for this reason, we will restrict our attention to the functors colim and lim. Examples will follow in the next subsection.

Definition 3.4. A model category structure on M^{J} is said to be:

- *Cofibrantly homotopy meaningful* if $\mathrm{colim}_{\mathsf{J}}$ preserves weak equivalences between cofibrant objects,
- *Fibrantly homotopy meaningful* if $\lim_{\mathsf{J}}$ preserves weak equivalences between fibrant objects.

3.3.4 Cofibrantly homotopy meaningful model category structures. Since there is an adjunction:

$$\operatorname{colim}_{\mathsf{J}} : \mathsf{M}^{\mathsf{J}} \rightleftarrows \mathsf{M} : \Delta_{\mathsf{J}},$$

for a model category structure on M^{J} to be cofibrantly homotopy meaningful, we need this adjunction to be a Quillen adjunction, i.e., we need Δ_{J} to preserve fibrations and acyclic fibrations. This leaves very little choice as to the model category structure of M^{J}. That is:

Lemma 3.10. *A model category structure on* M^{J} *is cofibrantly homotopy meaningful if:*

(i) *The weak equivalences are objectwise weak equivalences,*

(ii) *The fibrations are objectwise fibrations.*

In this case, the total left derived functor of $\operatorname{colim}_{\mathsf{J}}$ *exists and is termed the homotopy colimit:*

$$\operatorname{hocolim}_{\mathsf{J}} := \mathbb{L}\operatorname{colim}_{\mathsf{J}} : \mathsf{Ho}(\mathsf{M}^{\mathsf{J}}) \to \mathsf{Ho}(\mathsf{M}).$$

Moreover,

$$\operatorname{hocolim}_{\mathsf{J}}(X) \cong \operatorname{colim}_{\mathsf{J}}(X')$$

for any cofibrant object X' *weakly equivalent to* X*.*

3.3.5 Fibrantly homotopy meaningful model category structures. Since there is an adjunction:

$$\Delta_{\mathsf{J}} : \mathsf{M} \rightleftarrows \mathsf{M}^{\mathsf{J}} : \lim_{\mathsf{J}},$$

for a model category structure on M^{J} to be fibrantly homotopy meaningful, we need this adjunction to be a Quillen adjunction, i.e., we need Δ_{J} to preserve cofibrations and acyclic cofibrations. This, again, leaves very little choice as to the model category structure of M^{J}. That is:

Lemma 3.11. *A model category structure on* M^{J} *is fibrantly homotopy meaningful if:*

(i) *The weak equivalences are objectwise weak equivalences,*

(ii) *The cofibrations are objectwise cofibrations.*

In this case, the total right derived functor of $\lim_{\mathsf{J}}$ *exists and is termed the homotopy limit:*

$$\operatorname{holim}_{\mathsf{J}} := \mathbb{R}\lim_{\mathsf{J}} : \mathsf{Ho}(\mathsf{M}^{\mathsf{J}}) \to \mathsf{Ho}(\mathsf{M}).$$

Moreover,

$$\operatorname{holim}_{\mathsf{J}}(X) \cong \lim_{\mathsf{J}}(X')$$

for any fibrant object X' *weakly equivalent to* X*.*

3.3.d Homotopy Colimits and SSet

Let J be any small category. Dwyer and Kan (see [5] and [9]) have introduced a model structure on $\mathsf{SSet}^{\mathsf{J}}$ displaying the properties that:

(i) The weak equivalences are objectwise weak equivalences,

(ii) The fibrations are objectwise fibrations.

(iii) The cofibrations are morphisms with the LLP with respect to acyclic fibrations.

That is, they introduce a cofibrantly homotopy meaningful model structure on $\mathsf{SSet}^{\mathsf{J}}$; although in [9], they termed this model structure "natural with respect to J" since for a functor $F\colon \mathsf{J}' \to \mathsf{J}$, the induced functor $F^*: \mathsf{SSet}^{\mathsf{J}} \to \mathsf{SSet}^{\mathsf{J}'}$ preserves weak equivalences.

The dual to this construction also can be introduced so as to define a fibrantly homotopy meaningful model structure on $\mathsf{SSet}^{\mathsf{J}}$. We will focus on the former case rather than the latter.

For the cofibrantly homotopy meaningful model structure on $\mathsf{SSet}^{\mathsf{J}}$, homotopy colimits have been well-studied by Bousfield and Kan, see [5]. In fact, a very concrete method for computing them was introduced. As it sheds insight into homotopy colimits, we briefly review this construction. First, we must introduce the notion of a bisimplicial object.

3.3.6 Bisimplicial objects in a category. In a category C a *bisimplicial object* is a simplicial object in the category of simplicial objects over C. More concretely, a bisimplicial object is given by a sequence of objects $K_{m,n}$, for $m, n \geq 0$, as well as "vertical" and "horizontal" face and degeneracy morphisms:

$$\begin{array}{ccc} K_{m-1,n} & \xleftarrow{\partial_i^h} & K_{m,n} \\ \downarrow{\scriptstyle \partial_j^v} & & \downarrow{\scriptstyle \partial_j^v} \\ K_{m-1,n-1} & \xleftarrow{\partial_i^h} & K_{m,n-1} \end{array} \qquad\qquad \begin{array}{ccc} K_{m,n+1} & \xrightarrow{\sigma_i^h} & K_{m+1,n+1} \\ \uparrow{\scriptstyle \sigma_j^v} & & \uparrow{\scriptstyle \sigma_j^v} \\ K_{m,n} & \xrightarrow{\sigma_i^h} & K_{m+1,n} \end{array}$$

for $0 \leq i \leq n$ and $0 \leq j \leq m$ such that these diagrams commute and both ∂_j^v, σ_j^v and ∂_i^h, σ_i^h satisfy the simplicial identities given in (3.2).

Of special interest is the category of bisimplicial sets, $\mathsf{S}^2\mathsf{Set}$, and the category of bisimplicial abelian groups, $\mathsf{S}^2\mathsf{Ab}$.

3.3.7 The simplicial replacement functor. Given a functor $X : \mathsf{J} \to \mathsf{SSet}$ for a small category J, define the *simplicial replacement functor*

$$\amalg_{\mathsf{J}} : \mathsf{SSet}^{\mathsf{J}} \to \mathsf{S}^2\mathsf{Set}$$

by

$$\amalg_{\mathsf{J}}(X)_{m,n} = \coprod_{\alpha \in N\mathsf{J}_m} X(\mathsf{dom}(\alpha))_n.$$

The horizontal face morphisms, ∂_i^h, are given by requiring that

$$\begin{array}{ccc}
\coprod_{\alpha\in N\int_{m-1}} X(\mathsf{dom}(\alpha))_n & \xleftarrow{\quad\partial_i^h\quad} & \coprod_{\alpha\in N\int_{m}} X(\mathsf{dom}(\alpha))_n \\
\Big\uparrow \iota_{\partial_i^{N\int}\alpha} & & \Big\uparrow \iota_\alpha \\
X(\mathsf{dom}(\partial_i^{N\int}\alpha))_n & \xleftarrow{\left\{\begin{array}{lll} X(\alpha_1)_n & \text{if} & i=0 \\ \text{id} & \text{if} & 0<i\le n \end{array}\right\}} & X(\mathsf{dom}(\alpha))_n
\end{array}$$

commute. The degeneracy morphisms, σ_i^h, are given by the commuting diagram

$$\begin{array}{ccc}
\coprod_{\alpha\in N\int_{m}} X(\mathsf{dom}(\alpha))_n & \xrightarrow{\quad\sigma_i^h\quad} & \coprod_{\alpha\in N\int_{m+1}} X(\mathsf{dom}(\alpha))_n \\
\Big\uparrow \iota_\alpha & & \Big\uparrow \iota_{\sigma_i^{N\int}\alpha} \\
X(\mathsf{dom}(\alpha))_n & \xrightarrow{\quad\text{id}\quad} & X(\mathsf{dom}(\sigma_i^{N\int}\alpha))_n
\end{array}$$

The vertical face morphisms, ∂_i^v, and the degeneracy morphisms, σ_i^v, are given by requiring that the following diagrams

$$\begin{array}{ccc}
X(\mathsf{dom}(\alpha))_n & \xrightarrow{\iota_\alpha} & \coprod_{\alpha\in N\int_{m}} X(\mathsf{dom}(\alpha))_n \\
\Big\downarrow \partial_i^{X(\mathsf{dom}(\alpha))} & & \Big\downarrow \partial_i^v \\
X(\mathsf{dom}(\alpha))_{n-1} & \xrightarrow{\iota_\alpha} & \coprod_{\alpha\in N\int_{m}} X(\mathsf{dom}(\alpha))_{n-1}
\end{array}
\qquad
\begin{array}{ccc}
X(\mathsf{dom}(\alpha))_{n+1} & \xrightarrow{\iota_\alpha} & \coprod_{\alpha\in N\int_{m}} X(\mathsf{dom}(\alpha))_{n+1} \\
\Big\uparrow \sigma_i^{X(\mathsf{dom}(\alpha))} & & \Big\uparrow \sigma_i^v \\
X(\mathsf{dom}(\alpha))_{n} & \xrightarrow{\iota_\alpha} & \coprod_{\alpha\in N\int_{m}} X(\mathsf{dom}(\alpha))_{n}
\end{array}$$

commute. Here $\partial_i^{X(\mathsf{dom}(\alpha))}$ and $\sigma_i^{X(\mathsf{dom}(\alpha))}$ are the face and degeneracy morphisms of $X(\mathsf{dom}(\alpha))$.

3.3.8 The diagonal functor. The *diagonal functor* associates to a bisimplicial set a simplicial set, i.e.,

$$\text{diag} : \mathsf{S^2Set} \to \mathsf{SSet}.$$

For a bisimplicial set X,

$$\text{diag}(X)_n = X_{n,n},$$

and the face and degeneracy morphisms for diag(X) are given by $\partial_i = \partial_i^h \partial_i^v$ and $\sigma_i = \sigma_i^h \sigma_i^v$; the degeneracy morphisms can be visualized by considering the diagram

$$\begin{array}{ccc}
X_{m-1,n} & \xleftarrow{\partial_i^h} & X_{m,n} \\
\Big\downarrow \partial_i^v & \swarrow \partial_i & \Big\downarrow \partial_i^v \\
X_{m-1,n-1} & \xleftarrow{\partial_i^h} & X_{m,n-1}
\end{array}$$

hence the name "diagonal."

Using the simplicial replacement functor, one can explicitly define simplicial homotopy colimits.

Definition 3.5. The simplicial homotopy colimit:

$$\text{hocolim}_{\mathsf{J}}^{\mathsf{SSet}} : \mathsf{SSet}^{\mathsf{J}} \to \mathsf{SSet},$$

is given by the the composite:

$$\mathsf{SSet}^{\mathsf{J}} \xrightarrow{\amalg_{\mathsf{J}}} \mathsf{S}^2\mathsf{Set} \xrightarrow{\text{diag}} \mathsf{SSet}.$$

The usefulness of considering homotopy colimits is that one can make statements like:

Lemma 3.12. *Let $X, Y : \mathsf{J} \to \mathsf{SSet}$ are two functors, with J small. If $f : X \xrightarrow{\cdot} Y$ is objectwise a weak equivalence, then*

$$\text{hocolim}_{\mathsf{J}}^{\mathsf{SSet}}(f) : \text{hocolim}_{\mathsf{J}}^{\mathsf{SSet}}(X) \to \text{hocolim}_{\mathsf{J}}^{\mathsf{SSet}}(Y)$$

is a weak equivalence in SSet.

3.3.9 The Bousfield-Kan spectral sequence. Recall from Definition 3.2 that for a functor $L : \mathsf{J} \to \mathsf{Ab}$, we defined the homology of J with coefficients in L, denoted by $H_n(\mathsf{J}, L)$. Given a functor $X : \mathsf{J} \to \mathsf{SSet}$, we obtain a functor $H_p(X) : \mathsf{J} \to \mathsf{Ab}$ defined as the composite:

$$\mathsf{J} \xrightarrow{X} \mathsf{SSet} \xrightarrow{H_p} \mathsf{Ab}$$

and can thus consider the homology group $H_n(\mathsf{J}, H_p(X))$.

Spectral sequences provide a method for computing the homology group of a chain complex in terms of the homology of another chain complex, i.e., they capture the notion of "convergence in homology." We refer the reader to [6] and [28] for more on this subject. In [5] the following spectral sequence was introduced:

Theorem 3.4. *For $X : \mathsf{J} \to \mathsf{SSet}$ there is a spectral sequence:*

$$E^2_{p,q} = H_p(\mathsf{J}, H_q(X)) \Rightarrow H_{p+q}(\text{hocolim}_{\mathsf{J}}^{\mathsf{SSet}}(X)).$$

3.3.10 Topological homotopy colimits. Using the Quillen equivalence:

$$|-| : \mathsf{SSet} \rightleftarrows \mathsf{Top} : \text{sing},$$

one can use the formulation of homotopy colimits for simplicial sets to define topological homotopy colimits. That is, one can define the topological homotopy colimit to be the composite of:

$$\mathsf{Top}^{\mathsf{J}} \xrightarrow{\text{sing}_*} \mathsf{SSet}^{\mathsf{J}} \xrightarrow{\text{hocolim}_{\mathsf{J}}^{\mathsf{SSet}}} \mathsf{SSet} \xrightarrow{|-|} \mathsf{Top},$$

where sing_* is the functor on functor categories induced from the singular functor. Instead, we opt for the simpler approach of directly defining the homotopy colimit.

In [26], Vogt studies topological homotopy colimits and relates them to the constructions in [5]; in particular, he introduces an explicit formula for the topological homotopy colimit which will be useful for making calculations. For $n \geq 1$, let

$$\mathsf{Mor}_n(a,b) = \{(\alpha_n, \dots, \alpha_1) \in (\mathsf{Mor}(\mathsf{J}))^n : \alpha_n \circ \cdots \circ \alpha_1 : a \to b \text{ in } \mathsf{J}\},$$

where $\mathsf{Mor}(\mathsf{J})$ is the set of morphisms in J, and

$$\mathsf{Mor}_0(a,a) = \{\text{id}_a\}, \qquad \mathsf{Mor}_0(a,b) = \emptyset, \qquad a \neq b.$$

Then the topological homotopy colimit is given by

$$\text{hocolim}_{\mathsf{J}}^{\mathsf{Top}}(X) = \frac{\coprod_{a,b \in \mathsf{Ob}(\mathsf{J})} \coprod_{n=0}^{\infty} \mathsf{Mor}_n(a,b) \times I^n \times X(a)}{\sim}$$

where I is the unit interval and $\sim$ is the equivalence relation given by

$$(t_n, \alpha_n, \dots, t_1, \alpha_1; x) \sim \begin{cases} (t_n, \alpha_n, \dots, t_2, \alpha_2; x) & \alpha_1 = \text{id} \\ (t_n, \alpha_n, \dots, \alpha_{i+1}, t_i t_{i-1}, \alpha_{i-1}, \dots, \alpha_1; x) & \alpha_i = \text{id}, 1 < i \\ (t_n, \alpha_n, \dots, t_{i+1}, \alpha_{i+1} \circ \alpha_i, t_{i-1}, \dots, \alpha_1; x) & t_1 = 1,\ i < n \\ (t_{n-1}, \alpha_{n-1}, \dots, \alpha_1; x) & t_n = 1 \\ (t_n, \alpha_n, \dots, \alpha_{i+1}; X(\alpha_i \circ \cdots \circ \alpha_1)(x)) & t_i = 0 \end{cases}$$

There is again a spectral sequence (also termed the Bousfield-Kan spectral sequence) that allows one to compute the homology of the homotopy colimit.

Theorem 3.5. *For $X : \mathsf{J} \to \mathsf{Top}$ there is a spectral sequence:*

$$E^2_{p,q} = H_p(\mathsf{J}, H_q(X)) \Rightarrow H_{p+q}(\text{hocolim}_{\mathsf{J}}^{\mathsf{Top}}(X)).$$

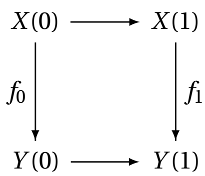
3.4 The Model Structure of Diagrams

One does not always have the luxury afforded in SSet and Top, i.e., for a general model category M, it is not known whether there exists a model structure on M^{J}. In this section, we determine conditions on a small category J so that there exists a cofibrantly homotopy meaningful model structure on M^{J}. Rather than proceeding directly to the most general result, we begin by considering a simple small category so as to illustrate the general principles involved.

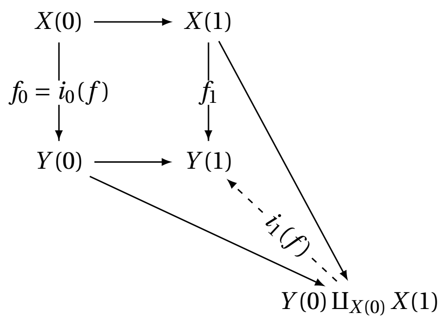
3.4.a The Simplest Nontrivial Diagram

In this subsection, we will consider small categories of shape $0 \to 1$, i.e., the simplest diagrams in M that are nontrivial (not discrete) and have no cycles. For diagrams of this form, we are able to define both cofibrantly and fibrantly homotopy meaningful model category structures. The more general constructions that we will later introduce can be intuitively understood through this simple example.

3.4.1 A cofibrantly homotopy meaningful model structure on $\mathsf{M}^{0\to 1}$. Let $f : X \dot{\to} Y$ be a morphism in $\mathsf{M}^{0\to 1}$, i.e., we have a commuting diagram:



This yields a commuting diagram:



Using the morphisms $i_0(f)$ and $i_1(f)$ there is the following:

Proposition 3.5. *The category $\mathsf{M}^{0\to 1}$ has a cofibrantly homotopy meaningful model category structure with the following choices of weak equivalences, fibrations and cofibrations: a morphism $f : X \dot{\to} Y$ is*

We. *A weak equivalence if f_0 and f_1 are weak equivalences in M,*

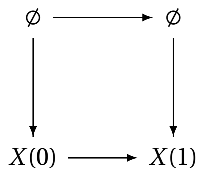
Fib. *A fibration if f_0 and f_1 are fibrations in M,*

Cof. *A cofibration if $i_0(f)$ and $i_1(f)$ are cofibrations in M.*

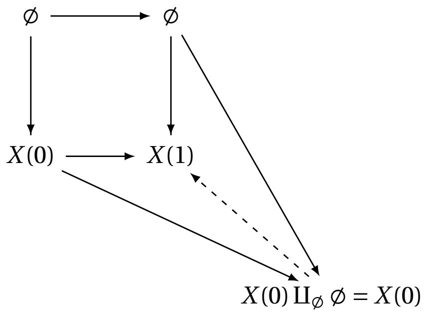
It is important to understand which objects are cofibrant given this model structure. Consider an object $X(0) \to X(1)$ in $\mathsf{M}^{0\to 1}$. This object is cofibrant in $\mathsf{M}^{0\to 1}$ if it is of the form:

$$X(0) \rightarrowtail X(1)$$

with $X(0)$ cofibrant. To see this, consider the commuting diagram:



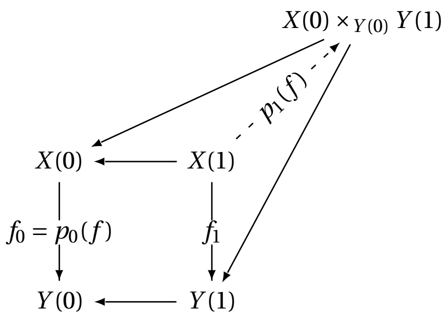
Then, this morphism in $\mathsf{M}^{0\to 1}$ is a cofibration if the dashed morphism in the following diagram



is a cofibration in M, but this morphism is exactly the morphism $X(0) \to X(1)$.

3.4.2 A fibrantly homotopy meaningful model structure on $\mathsf{M}^{0\to 1}$.

The dual of Proposition 3.5 can be given. In this case we consider the commuting diagram:



and we have the following fibrantly homotopy meaningful model category structure on $\mathsf{M}^{0\to 1}$.

Proposition 3.6. *The category $\mathsf{M}^{0\to 1}$ has a fibrantly homotopy meaningful model category structure with the following choices of weak equivalences, fibrations and cofibrations: a morphism $f : X \stackrel{\bullet}{\to} Y$ is*

We. *A weak equivalence if f_0 and f_1 are weak equivalences in M,*

Fib. *A fibration if $p_0(f)$ and $p_1(f)$ are fibrations in M.*

Cof. *A cofibration if f_0 and f_1 are cofibrations in M,*

3.4.b Direct Categories

We now introduce a result that will be central in defining fiberwise model structures for categories of hybrid objects—it allows one to define a cofibrantly homotopy meaningful model structure on M^{J} for certain small categories J.

3.4.3 Direct categories. To every natural number n, there is an associated category $\mathbf{n}$ with $\mathsf{Ob}(\mathbf{n}) = \{0,\dots,n-1\}$ and for every two objects i and j of $\mathbf{n}$, there is one morphism between these objects iff $i \leq j$.

Definition 3.6. A functor $\deg : \mathsf{J} \to \mathbf{n}$ is called a *linear extension* if the image of a non-identity morphism is a non-identity morphism; in this case J is said to have degree $\mathbf{n}$.

Definition 3.7. A small category J is a *direct category* if there is a linear extension $\deg : \mathsf{J} \to \mathbf{n}$.

3.4.4 A cofibrantly homotopy meaningful model category structure on M^{J}. In the case when J is a direct category, M^{J} can be given a cofibrantly homotopy meaningful model category structure (as well as a fibrantly homotopy meaningful model structure although this will not be discussed). Moreover, this model category structure directly generalizes the one previously given. Our introduction of this construction follows from [17].

We begin by considering the overcategory $(\mathsf{J} \downarrow a)$ for an object a of J. Consider the subcategory J_a of this category given by removing the object (id_a, a), i.e., it is the category of all nonidentity morphisms with codomain a. Specifically, the objects of J_a are pairs (f, b) where $f : b \to a$ and $b \neq a$. A morphism $h : (f, b) \to (g, c)$ is a morphism $h : b \to c$ such that $g \circ h = f$.

There is a functor $S_a : \mathsf{J}_a \to \mathsf{J}$ defined on objects by $S_a(f, b) = b$ and on morphisms by $S_a(h) = h$. This induces a functor $S_a^* : \mathsf{M}^{\mathsf{J}} \to \mathsf{M}^{\mathsf{J}_a}$ with $S_a^*(X) = X \circ S_a$. Define the *latching space functor* $L_a : \mathsf{M}^{\mathsf{J}} \to \mathsf{M}$ as the composite:

$$\mathsf{M}^{\mathsf{J}} \xrightarrow{S_a^*} \mathsf{M}^{\mathsf{J}_a} \xrightarrow{\operatorname{colim}_{\mathsf{J}_a}} \mathsf{M}.$$

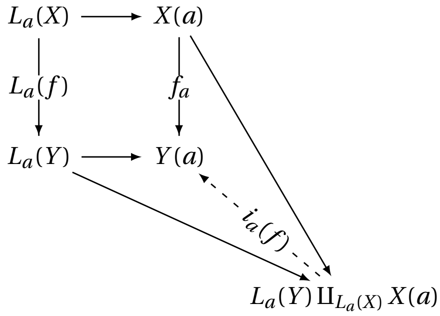
Now, let $f : X \stackrel{\bullet}{\to} Y$ in M^{J}. Then for every object a of J, we have a commuting diagram:

$$\begin{array}{ccc} S_a^*(X) & \longrightarrow & \Delta_{\mathsf{J}_a}(X(a)) \\ {\scriptstyle S_a^*(f)}\downarrow & & \downarrow{\scriptstyle \Delta_{\mathsf{J}_a}(f_a)} \\ S_a^*(Y) & \longrightarrow & \Delta_{\mathsf{J}_a}(Y(a)) \end{array}$$

which yields a commuting diagram:

$$\begin{array}{ccc} L_a(X) & \longrightarrow & X(a) \\ {\scriptstyle L_a(f)}\downarrow & & \downarrow{\scriptstyle f_a} \\ L_a(Y) & \longrightarrow & Y(a) \end{array}$$

Corresponding to this diagram, we have a morphism $i_a(f)$ defined uniquely as follows:



With this construction in hand, there is the following important theorem (see [17]).

Theorem 3.6. *For J a direct category, the category M^{J} has a cofibrantly homotopy meaningful model category structure with the following choices of weak equivalences, fibrations and cofibrations: a morphism $f : X \dot{\to} Y$ is*

We. *A weak equivalence if f is objectwise a weak equivalence in M,*

Fib. *A fibration if f is objectwise a fibration in M,*

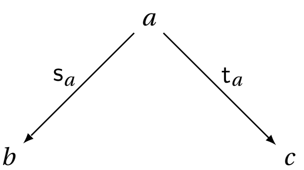
Cof. *A cofibration if $i_a(f) : L_a(Y) \amalg_{L_a(X)} X(a) \to Y(a)$ is a cofibration in M for all objects a of J.*

3.5 Fiberwise Hybrid Homotopy Theory

The motivation for the title of this section, "fiberwise hybrid homotopy theory," is that, using the previous constructions relating to the model structure of diagrams, we can construct a model structure on fibers of $\mathsf{Hy}(\mathsf{M})$, $\mathsf{Hy}(\mathsf{M})_{\mathscr{D}} \cong \mathsf{M}^{\mathscr{D}}$, for every D-category $\mathscr{D}$. This allows us to relate the model structure on $\mathsf{M}^{\mathscr{D}}$ to the model structure on M. For example, given a hybrid object $\mathbf{X} : \mathscr{D} \to \mathsf{M}$, we can take its homotopy colimit to obtain an object of M that is well-behaved with respect to weak equivalences. Applications of these ideas are discussed in the context of topology and homology.

3.5.a The Model Structure of $\mathsf{M}^{\mathscr{D}}$

We begin by using Theorem 3.6 to find a cofibrantly homotopy meaningful model structure on the fibers of $\mathsf{Hy}(\mathsf{M})$. The motivation for considering cofibrantly homotopy meaningful model structures is that, again, the basic diagram in a D-category is of the form:



for $a \in \mathsf{E}(\mathscr{D})$ and $b, c \in \mathsf{V}(\mathscr{D})$. That is, the structure of D-categories imply that we would like a model structure on $\mathsf{M}^{\mathscr{D}}$ that admits homotopy colimits.

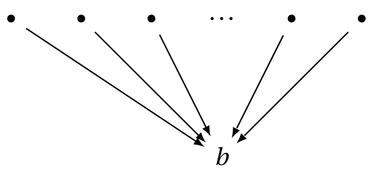
3.5.1 The degree functor. Recall that the category **2** consists of two objects and a single (non-identity) morphism: $0 \to 1$. Define the degree functor $\deg : \mathscr{D} \to \mathbf{2}$ on objects $a \in \mathscr{D}$ by

$$\deg(a) = \begin{cases} 0 & \text{if} \quad a \in \mathsf{E}(\mathscr{D}) \\ 1 & \text{if} \quad a \in \mathsf{V}(\mathscr{D}) \end{cases}$$

This functor sends every (non-identity) morphism in $\mathscr{D}$ to the single (non-identity) morphism in **2**. Since deg is thus a linear extension:

Lemma 3.13. *Every D-category $\mathscr{D}$ is a direct category.*

3.5.2 The latching space functor. We now produce the latching space functor for D-categories. Note that the axioms of a D-category imply that $\mathscr{D}_a$ is a discrete category for every object a of $\mathscr{D}$; in fact, $\mathscr{D}_a = \emptyset$ if $a \in \mathsf{E}(\mathscr{D})$. In the case when $b \in \mathsf{V}(\mathscr{D})$, $\mathscr{D}_b$ is the category of all arrows pointing to b, i.e. it can be visualized as follows:

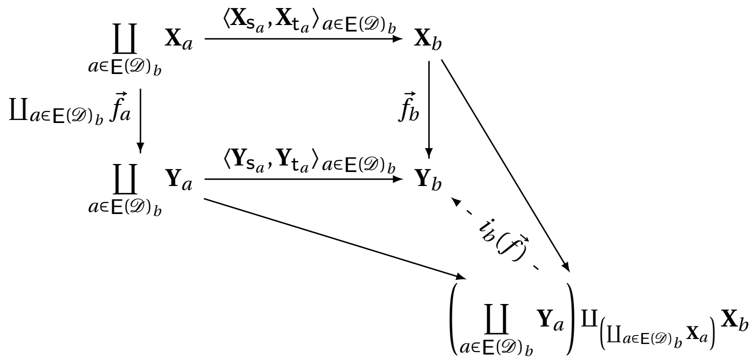


where all the objects $\bullet$ are in $\mathsf{E}(\mathscr{D})$. Therefore, for $b \in \mathsf{V}(\mathscr{D})$, define

$$\mathsf{E}(\mathscr{D})_b = \{a \in \mathsf{E}(\mathscr{D}) : \exists\ \alpha : a \to b \text{ in } \mathscr{D}\} = \{a \in \mathsf{E}(\mathscr{D}) : b = \mathsf{cod}(\mathsf{s}_a) \text{ or } b = \mathsf{cod}(\mathsf{t}_a)\}.$$

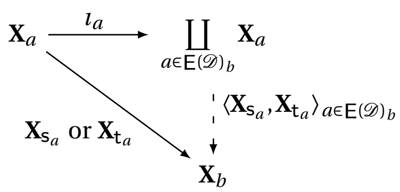
Let $\vec{f} : \mathbf{X} \dot{\to} \mathbf{Y}$ in $\mathsf{M}^{\mathscr{D}}$. Using the construction in the previous subsection (see Paragraph 3.4.4), we define the following morphisms:

- For every $a \in \mathsf{E}(\mathscr{D})$, define $i_a(\vec{f}) = \vec{f}_a$.
- For every $b \in \mathsf{V}(\mathscr{D})$, define $i_b(\vec{f})$ to be the unique morphism induced by the following pushout diagram:



(3.5)

where $\langle \mathbf{X}_{\mathsf{s}_a}, \mathbf{X}_{\mathsf{t}_a}\rangle_{a\in\mathsf{E}(\mathscr{D})_b}$ and $\langle \mathbf{Y}_{\mathsf{s}_a}, \mathbf{Y}_{\mathsf{t}_a}\rangle_{a\in\mathsf{E}(\mathscr{D})_b}$ are the unique morphisms induced by the coproduct; for example, $\langle \mathbf{X}_{\mathsf{s}_a}, \mathbf{X}_{\mathsf{t}_a}\rangle_{a\in\mathsf{E}(\mathscr{D})_b}$ is the unique morphism making the following diagram:



commute.

Using these definitions, we have the following theorem which is a corollary of Theorem 3.6.

Theorem 3.7. *For any D-category $\mathscr{D}$, the category $\mathsf{M}^{\mathscr{D}}$ has a cofibrantly homotopy meaningful model category structure for the following choices of weak equivalences, fibrations and cofibrations: a morphism $\vec{f} : \mathbf{X} \dot{\to} \mathbf{Y}$ is*

We. *A weak equivalence if $\vec{f}$ is objectwise a weak equivalence in M,*

Fib. *A fibration if $\vec{f}$ is objectwise a fibration in M,*

Cof. *A cofibration if $i_a(\vec{f})$ and $i_b(\vec{f})$ are cofibrations in M for all $a \in \mathsf{E}(\mathscr{D})$ and $b \in \mathsf{V}(\mathscr{D})$.*

The importance of this theorem is that the model structure on $\mathsf{M}^{\mathscr{D}}$ was defined in such a way that homotopy colimits exist. That is, we have the following:

Corollary 3.3. *For every model category* M *and D-category* $\mathscr{D}$*,*

$$\operatorname{hocolim}_{\mathscr{D}} : \mathsf{Ho}(\mathsf{M}^{\mathscr{D}}) \to \mathsf{Ho}(\mathsf{M})$$

exists and

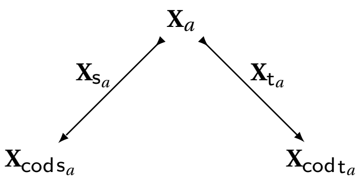
- $\operatorname{hocolim}_{\mathscr{D}}(\mathbf{X}) \cong \operatorname{hocolim}_{\mathscr{D}}(\mathbf{Y})$ *if* $\mathbf{X}$ *and* $\mathbf{Y}$ *are weakly equivalent.*
- $\operatorname{hocolim}_{\mathscr{D}}(\mathbf{X}) \cong \operatorname{colim}_{\mathscr{D}}(\mathbf{X}')$ *for every cofibrant hybrid object* $\mathbf{X}'$ *weakly equivalent to* $\mathbf{X}$*.*

This corollary implies that in order to compute homotopy colimits, we must first understand what the cofibrant objects are in $\mathsf{M}^{\mathscr{D}}$. This motivates the following:

Proposition 3.7. *For every D-category* $\mathscr{D}$*, an object* $\mathbf{X} : \mathscr{D} \to \mathsf{M}$ *of* $\mathsf{M}^{\mathscr{D}}$ *is cofibrant if for every* $a \in \mathsf{E}(\mathscr{D})$*:*

- $\mathbf{X}_a$ *is cofibrant,*
- $\mathbf{X}_{\mathsf{s}_a}$ *and* $\mathbf{X}_{\mathsf{t}_a}$ *are cofibrations.*

This proposition implies that $\mathbf{X}$ is cofibrant if for every $a \in \mathsf{E}(\mathscr{D})$, the corresponding diagram has the form:

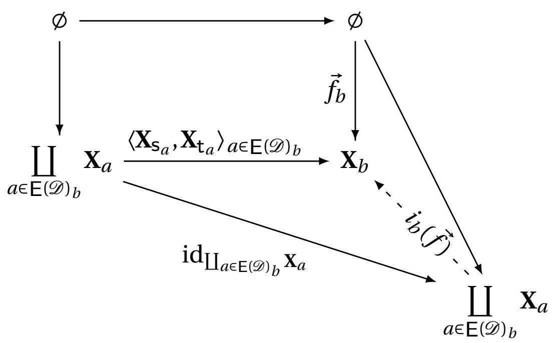


with $\mathbf{X}_a$ cofibrant.

Proof. The initial object in $\mathsf{M}^{\mathscr{D}}$ is given by $\Delta_{\mathscr{D}}(\emptyset)$, where $\emptyset$ is the initial object of M. Consider a morphism $\vec{f} : \Delta_{\mathscr{D}}(\emptyset) \xrightarrow{\cdot} \mathbf{X}$. Clearly, for $a \in \mathsf{E}(\mathscr{D})$,

$$i_a(\vec{f}) = \vec{f}_a : \emptyset \to \mathbf{X}_a$$

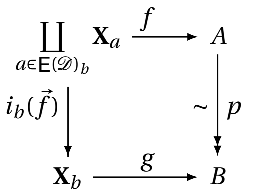
is a cofibration since $\mathbf{X}_a$ is cofibrant. For $b \in \mathsf{V}(\mathscr{D})$, the diagram in (3.5), becomes:



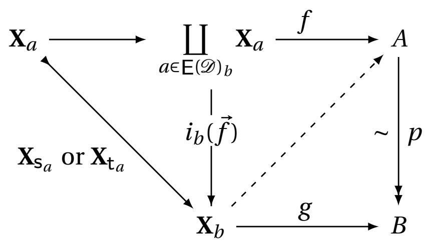
Therefore, for all $b \in \mathsf{V}(\mathscr{D})$,

$$i_b(\vec{f}) = \langle \mathbf{X}_{\mathsf{s}_a}, \mathbf{X}_{\mathsf{t}_a} \rangle_{a \in \mathsf{E}(\mathscr{D})_b}.$$

To verify that this is a cofibration, we utilize Proposition 3.1. That is, we show that $i_b(\vec{f})$ has the LLP with respect to acyclic fibrations. Consider a commuting diagram:

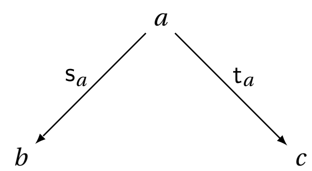


where p is an acyclic fibration. For every $b \in \mathsf{E}(\mathscr{D})_a$ there is an associated diagram



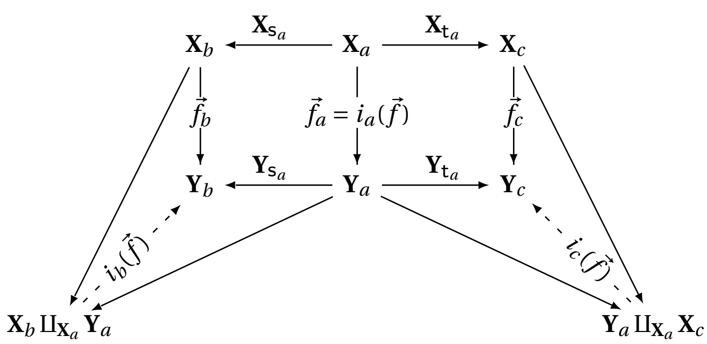
with the far right arrow either $\mathbf{X}_{\mathsf{s}_a}$ or $\mathbf{X}_{\mathsf{t}_a}$. In both cases, the dashed arrow exists by the assumption that these morphisms are cofibrations. Therefore, since this holds for all $a \in \mathsf{E}(\mathscr{D})_b$, the dashed arrow provides the desired lift for $i_b(\vec{f})$. □

3.5.3 Homotopy pushouts. To provide a specific application of the above theorem, we will discuss the corresponding constructions in the context of *homotopy pushouts*. This amounts to considering D-categories of a very specific form. Specifically, for the rest of this subsection we assume that

$\mathscr{D} =$ 

We apply Theorem 3.7 to obtain a model structure on $\mathsf{M}^{b\leftarrow a\rightarrow c}$; this is a well-known result and is discussed in detail in [10].

Let $\vec{f} : \mathbf{X} \dot{\rightarrow} \mathbf{Y}$ be a morphism in $\mathsf{M}^{b\leftarrow a\rightarrow c}$. Since $\mathsf{E}(\mathscr{D})_b = \mathsf{E}(\mathscr{D})_c = \{a\}$, the diagram in (3.5) becomes:



Therefore, we have the following:

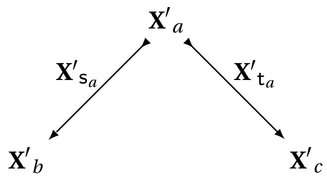
Corollary 3.4. *The category $\mathsf{M}^{b\leftarrow a\rightarrow c}$ has a cofibrantly homotopy meaningful model category structure with the following choices of weak equivalences, fibrations and cofibrations: a morphism $\vec{f}:\mathbf{X}\dot{\rightarrow}\mathbf{Y}$ is*

We. *A weak equivalence if $\vec{f}_a$, $\vec{f}_b$ and $\vec{f}_c$ are weak equivalences in M,*

Fib. *A fibration if $\vec{f}_a$, $\vec{f}_b$ and $\vec{f}_c$ are fibrations in M,*

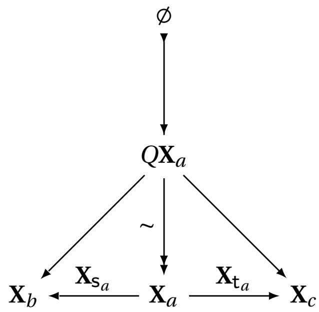
Cof. *A cofibration if $i_a(\vec{f})$, $i_b(\vec{f})$ and $i_c(\vec{f})$ are cofibrations in M.*

3.5.4 Computing homotopy pushouts. An important aspect of computing homotopy colimits is being able to, given an object $\mathbf{X}$ of $\mathsf{M}^{\mathscr{D}}$, compute a cofibrant object $\mathbf{X}'$ weakly equivalent to $\mathbf{X}$. When $\mathscr{D} = b \leftarrow a \rightarrow c$ this is a simple matter—for more general D-categories things become more complicated. In the simple setting we are considering here, the goal is to find a hybrid object $\mathbf{X}'$ of the form:

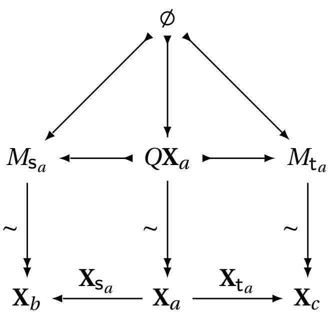


with $\mathbf{X}_a$ cofibrant; proposition 3.7 implies that this is a cofibrant object in $\mathsf{M}^{b\leftarrow a\rightarrow c}$.

If $Q\mathbf{X}_a$ is a cofibrant replacement of $\mathbf{X}_a$, then we have a commuting diagram:

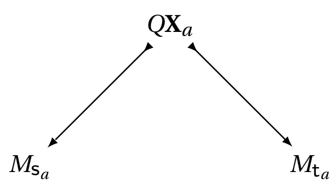


Factoring the left and right diagonal morphisms yields a commuting diagram:



Therefore, let

$\mathbf{X}'(b \leftarrow a \rightarrow c) :=$



which is a cofibrant object in $\mathsf{M}^{b\leftarrow a\rightarrow c}$ weakly equivalent to $\mathbf{X}$. Therefore, the homotopy colimit of $\mathbf{X}$, termed the *homotopy pushout*, is given by:

$$\mathrm{hocolim}_{b\leftarrow a\rightarrow c}(\mathbf{X}) \cong M_{\mathsf{s}_a} \amalg_{Q\mathbf{X}_a} M_{\mathsf{t}_a}.$$

3.5.b Hybrid Topology

Let $\mathbf{X}: \mathscr{D} \to \mathsf{Top}$ be a hybrid topological space. The goal is to use the constructions introduced up to this point, and especially those given in Paragraph 3.3.10, to associate to this data a single topological space (these constructions were first utilized in the context of hybrid systems in [3]).

If one were to take a naive approach to the problem of associating a topological space to a hybrid topological space, one would logically take the colimit of this diagram:

$$\mathrm{colim}_{\mathscr{D}}(\mathbf{X}) \quad = \quad \frac{\coprod_{a\in \mathsf{Ob}(\mathscr{D})} \mathbf{X}_a}{x \sim \mathbf{X}_\alpha(x) \qquad \alpha \in \mathsf{Mor}(\mathscr{D})}.$$

This is, in fact, the construction that has been applied to "hybrid data" in the past, namely in [23], although it was not recognized that this was actually the colimit as the categorical definition of hybrid systems was not available, i.e., the *hybrifold* was defined as $\mathrm{colim}_{\mathscr{D}}(\mathbf{X})$. The key point is that although this construction is the obvious way of associating a single space to a hybrid system, it is not the "correct" one. Allen Hatcher describes this aptly in [15], to quote:

> "It can easily happen that the [colimit] is rather useless because so much collapsing has occurred that little of the original diagram remains."

The framework of model categories allows us to conclude that the correct way to associate to a hybrid topological space a single topological space is through the use of homotopy colimits. In the case of categories of hybrid objects, and in the context of topological spaces, these have a particularly simple form.

Theorem 3.8. *For a hybrid topological space* $\mathbf{X}: \mathscr{D} \to \mathsf{Top}$,

$$\mathrm{hocolim}^{\mathsf{Top}}_{\mathscr{D}}(\mathbf{X}) = \frac{\left(\coprod_{b\in \mathsf{V}(\mathscr{D})} \mathbf{X}_b\right) \amalg \left(\coprod_{a\in \mathsf{E}(\mathscr{D})} (\mathbf{X}_a \times I)\right)}{(x,0) \sim \mathbf{X}_{\mathsf{s}_a}(x), \quad (x,1) \sim \mathbf{X}_{\mathsf{t}_a}(x), \qquad a \in \mathsf{E}(\mathscr{D})}.$$

Proof. Due to the first axiom, **AD1**, of a D-category, the formula for the topological homotopy colimit given in 3.3.10 becomes:

$$\begin{aligned}
\mathrm{hocolim}^{\mathsf{Top}}_{\mathscr{D}}(\mathbf{X}) \quad &= \quad \frac{\left(\coprod_{a\in \mathsf{Ob}(\mathscr{D})} \mathbf{X}_a\right) \amalg \left(\coprod_{a,b\in \mathsf{Ob}(\mathscr{D})} \mathsf{Mor}_{\not{\mathrm{id}}}(a,b) \times I \times \mathbf{X}_a\right)}{(1,\alpha_1;x) \sim x, \qquad (0,\alpha_1;x) \sim \mathbf{X}_{\alpha_1}(x)} \\
&= \quad \frac{\left(\coprod_{a\in \mathsf{Ob}(\mathscr{D})} \mathbf{X}_a\right) \amalg \left(\coprod_{(a,b)\in \mathsf{E}(\mathscr{D})\times \mathsf{V}(\mathscr{D})} \mathsf{Mor}_{\not{\mathrm{id}}}(a,b) \times I \times \mathbf{X}_a\right)}{(1,\alpha_1;x) \sim x, \qquad (0,\alpha_1;x) \sim \mathbf{X}_{\alpha_1}(x)} \\
&= \quad \frac{\left(\coprod_{a\in \mathsf{Ob}(\mathscr{D})} \mathbf{X}_a\right) \amalg \left(\coprod_{a\in \mathsf{E}(\mathscr{D})} \{\mathsf{s}_a\} \times I \times \mathbf{X}_a\right) \amalg \left(\coprod_{a\in \mathsf{E}(\mathscr{D})} \{\mathsf{t}_a\} \times I \times \mathbf{X}_a\right)}{(1,\mathsf{s}_a;x) \sim x \sim (1,\mathsf{t}_a;x), \quad (0,\mathsf{s}_a;x) \sim \mathbf{X}_{\mathsf{s}_a}(x), \quad (0,\mathsf{t}_a;x) \sim \mathbf{X}_{\mathsf{t}_a}(x)} \\
&\simeq \quad \frac{\left(\coprod_{b\in \mathsf{V}(\mathscr{D})} \mathbf{X}_b\right) \amalg \left(\coprod_{a\in \mathsf{E}(\mathscr{D})} I \times \mathbf{X}_a\right)}{(0,x) \sim \mathbf{X}_{\mathsf{s}_a}(x), \quad (1,x) \sim \mathbf{X}_{\mathsf{t}_a}(x)}
\end{aligned}$$

where the last equality is actually a homotopy equivalence essentially given by contracting the interval $[0,2]$ to the unit interval I. □

Definition 3.8. For a hybrid topological space $\mathbf{X} : \mathscr{D} \to \mathsf{Top}$, define its *underlying topological space* as:

$$\mathrm{top}(\mathscr{D}, \mathbf{X}) := \mathrm{hocolim}_{\mathscr{D}}^{\mathsf{Top}}(\mathbf{X}).$$

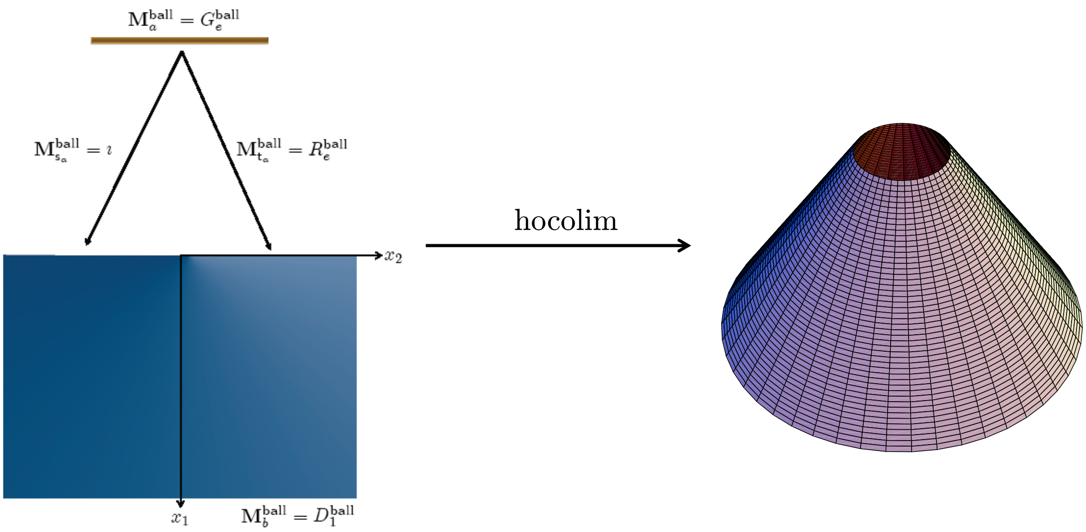


Figure 3.1: The underlying topological space of the bouncing ball.

Example 3.2. Recall that in Example 1.4 we introduced the hybrid manifold $\mathbf{M}^{\mathrm{ball}} : \mathscr{D}^{\mathrm{ball}} \to \mathsf{Man}$ associated to the bouncing ball. Let $\mathbf{X}^{\mathrm{ball}} : \mathscr{D}^{\mathrm{ball}} \to \mathsf{Top}$ be the hybrid topological space obtained from this hybrid manifold by forgetting about its smooth structure. The underlying topological space of the bouncing ball, $\mathrm{top}(\mathscr{D}^{\mathrm{ball}}, \mathbf{X}^{\mathrm{ball}})$, is therefore homotopic to the punctured cone; see Figure 3.1. Note that if one were to take the colimit and not the homotopy colimit, the result would be a cone.

3.5.c Hybrid Homology

The goal of this subsection is to understand the homology of a hybrid topological space $\mathbf{X} : \mathscr{D} \to \mathsf{Top}$ through the use of the Bousfield-Kan spectral sequence. First, we introduce:

3.5.5 The normalized Moore complex. For a simplicial abelian group A, there is a subcomplex of the Moore complex called the normalized complex (cf. [28]) and denoted by $N_\bullet(A)$ (not to be confused with the nerve). This is a chain complex with

$$N_n(A) = \bigcap_{i=0}^{n-1} \mathrm{Ker}(\partial_i : A_n \to A_{n-1})$$

and differential $d = (-1)^n \partial_n$. The normalized complex is important because of its relation to the "degenerate" subcomplex of the Moore complex; this is a complex $D_\bullet(A)$ with

$$D_n(A) = \bigoplus_{0 \le i \le n-1} \sigma_i(A_{n-1}).$$

These two complexes are related by the fact that $N_\bullet(A) \cong A_\bullet / D_\bullet(A)$. Even more important is the fact that there is an isomorphism

$$H_n(N_\bullet(A)) \cong H_n(A).$$

A proof of these statements can be found in [28].

In the case when we are considering the simplicial abelian group J^L associated to a functor $L : \mathsf{J} \to \mathsf{Ab}$, we denote the normalized complex by $N_\bullet(\mathsf{J}^L)$ and the degenerate complex by $D_\bullet(\mathsf{J}^L)$. With this notation the above statements imply that we have the isomorphisms

$$N_\bullet(\mathsf{J}^L) \cong \frac{\mathsf{J}^L_\bullet}{D_\bullet(\mathsf{J}^L)}, \qquad H_n(N_\bullet(\mathsf{J}^L)) \cong H_n(\mathsf{J}, L).$$

For a hybrid abelian group $\mathbf{L} : \mathscr{D} \to \mathsf{Ab}$, by the axioms of a D-category and specifically **AD1**, it follows that $\mathscr{D}^{\mathbf{L}}_n \cong D_n(\mathscr{D}^{\mathbf{L}})$ for all $n \geq 2$. Therefore, we have the following:

Lemma 3.14. *If $\mathscr{D}$ is a D-category, then for any hybrid abelian group $\mathbf{L} : \mathscr{D} \to \mathsf{Ab}$, $H_n(\mathscr{D}, \mathbf{L}) \cong 0$ for $n \geq 2$.*

3.5.6 Hybrid homology. The *hybrid homology* of a hybrid topological space is defined to be the homology of its underlying topological space:

$$HY_n(\mathscr{D}, \mathbf{X}) := H_n(\mathrm{top}(\mathscr{D}, \mathbf{X})) = H_n(\mathrm{hocolim}^{\mathsf{Top}}_{\mathscr{D}}(\mathbf{X})).$$

Lemma 3.14 implies that for a hybrid topological space $\mathbf{X} : \mathscr{D} \to \mathsf{Top}$, the Bousfield-Kan spectral sequence

$$E^2_{p,q} = H_p(\mathscr{D}, H_q(\mathbf{X})) \Rightarrow HY_n(\mathscr{D}, \mathbf{X})$$

consists of two columns, $p = 0$ and $p = 1$. Under these circumstances (see [28], and especially the corresponding errata) it follows that we have established the following:

Theorem 3.9. *For a hybrid topological space $\mathbf{X} : \mathscr{D} \to \mathsf{Top}$, there are short exact sequences*

$$0 \longrightarrow H_0(\mathscr{D}, H_n(\mathbf{X})) \longrightarrow HY_n(\mathscr{D}, \mathbf{X}) \longrightarrow H_1(\mathscr{D}, H_{n-1}(\mathbf{X})) \longrightarrow 0$$

for all $n \geq 0$.

3.5.7 Hybrid homology and graph homology. To better understand the preceding theorem, let us restrict our attention to a very special case.

A hybrid topological space $\mathbf{X} : \mathscr{D} \to \mathsf{Top}$ is said to be *domain contractible* if there is a weak equivalence in $\mathsf{Top}^{\mathscr{D}}$:

$$\mathbf{X} \stackrel{\cdot}{\to} \Delta_{\mathscr{D}}(*)$$

where $*$ is a point; this implies, for example, that $H_0(\mathbf{X}_a) \cong \mathbb{Z}$ and $H_n(\mathbf{X}_a) \cong 0$, $n \geq 1$, for every $a \in \mathsf{Ob}(\mathscr{D})$. It is easy to verify the following corollary of Theorem 3.9.

Corollary 3.5. *Assume that $\mathscr{D}$ is finite, i.e., has a finite number of objects (and hence morphisms), and that* $\mathbf{X} : \mathscr{D} \to \mathsf{Top}$ *is domain contractible. Then*

$$HY_n(\mathscr{D}, \mathbf{X}) \cong H_n(\mathrm{grph}(\mathscr{D}))$$

for all $n \geq 0$; here $H_n(\mathrm{grph}(\mathscr{D}))$ is the homology of the graph $\mathrm{grph}(\mathscr{D})$.

This rather obvious statement has some interesting connotations. Namely, it indicates that the formulation of hybrid homology is in fact the right one; it agrees with the homology of the discrete structure of a hybrid topological space when the continuous data is trivial.

Example 3.3. For the hybrid topological space $\mathbf{X}^{\mathrm{ball}} : \mathscr{D}^{\mathrm{ball}} \to \mathsf{Top}$, the hybrid homology is given by:

$$HY_1(\mathscr{D}^{\mathrm{ball}}, \mathbf{X}^{\mathrm{ball}}) \cong HY_0(\mathscr{D}^{\mathrm{ball}}, \mathbf{X}^{\mathrm{ball}}) \cong \mathbb{Z}.$$

Note that if one were to consider the homology of the colimit (and not the homotopy colimit), the first homology would be trivial. One concludes that the colimit does not yield the right topology, while the homotopy colimit does.

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