Lyapunov Theory for Zeno Stability

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***Abstract*—Zeno behavior is a dynamic phenomenon unique to hybrid systems in which an infinite number of discrete transitions occurs in a finite amount of time. This behavior commonly arises in mechanical systems undergoing impacts and optimal control problems, but its characterization for general hybrid systems is not completely understood. The goal of this paper is to develop a stability theory for Zeno hybrid systems that parallels classical Lyapunov theory; that is, we present Lyapunov-like sufficient conditions for Zeno behavior obtained by mapping solutions of complex hybrid systems to solutions of simpler Zeno hybrid systems defined on the first quadrant of the plane. These conditions are applied to Lagrangian hybrid systems, which model mechanical systems undergoing impacts, yielding simple sufficient conditions for Zeno behavior. Finally, the results are applied to robotic bipedal walking.**

I. Introduction

Zeno behavior occurs in hybrid systems when an execution (or solution) undergoes infinitely many discrete transitions in a finite amount of time. Prior to the introduction of hybrid systems, and in contrast with the view of Zeno behavior as a modeling pathology, Zeno phenomena have long been studied in the fields of nonsmooth mechanics and optimal control. While understanding of Zeno behavior is, in these domains, quite sophisticated, many basic problems in the theory of Zeno behavior of general hybrid systems remain unsolved.

This paper studies the connections between Zeno behavior and Lyapunov stability. In classical dynamical systems, stability is a property of the asymptotic behavior of trajectories as time goes to infinity. In the model of hybrid systems used in this paper, time is measured with two variables, one for real time and the other for the number of discrete transitions. Zeno stability is a hybrid analog of classical stability in the case that real time stays bounded, while the number of discrete transitions approaches infinity.

A. Summary of Contributions

The contributions of this paper are: 1) a theorem connecting asymptotic Zeno stability and the geometry of Zeno equilibria; 2) Lyapunov-like sufficient conditions for local Zeno stability for hybrid systems over cycles; 3) easily verifiable sufficient conditions for Zeno stability of Lagrangian hybrid systems, which model mechanical systems undergoing impacts.

Our first contribution deals with geometry of special invariant sets, termed Zeno equilibria, which are analogous to equilibrium points of dynamical systems. A Zeno equilibrium is a set of points (with one point in each discrete domain) that is invariant under the discrete dynamics of the hybrid system but not the continuous dynamics. Our result shows that a Zeno equilibrium is asymptotically Zeno stable if and only if it is isolated (each point in each domain is isolated). This result clarifies the limitations of existing results, [1], [2], [3], [4], [5], which focus either on isolated Zeno equilibria or asymptotic convergence. In particular, their application to Lagrangian systems with impacts (such as bouncing balls) is restricted to systems with one-dimensional configuration manifolds, as higher-dimensional systems cannot have isolated Zeno equilibria [2].

The next contribution, which is the main result of the paper, consists of Lyapunov-like sufficient conditions for Zeno stability that, in contrast to existing results, apply to both isolated and non-isolated Zeno equilibria. The classical Lyapunov theorem uses a Lyapunov function to map solutions of a complex differential equation down to the solution of a simple one-dimensional differential inclusion, and then uses the structure of the Lyapunov function to prove that the original system inherits the stability properties of the one-dimensional system. Our approach to Zeno stability is similarly inspired. We use Lyapunov-like functions to map executions of a complex hybrid systems down to executions of simple two-dimensional differential inclusion hybrid systems, and then use the structure of the Lyapunov-like functions to prove that the original system inherits Zeno stability properties of the two-dimensional system.

Our final contribution applies the Lyapunov-like theorem to Lagrangian hybrid systems (which model mechanical systems undergoing impacts). Note that our Lyapunov-like theorem can be applied to Lagrangian hybrid systems precisely because it applies equally well to isolated and non-isolated Zeno equilibria. While the technical machinery of hybrid systems is not needed to develop the theory of mechanical systems with impacts, we feel that any reasonable stability theory of hybrid systems ought to cover this important special case. To prove Zeno stability in Lagrangian hybrid systems, we give a general form for a Lyapunov-like function that applies to any Lagrangian hybrid system whose vector field satisfies simple algebraic conditions at a single point (based upon the *unilateral constraint function* defining the discrete component of the Lagrangian hybrid system). Finally, it is shown how the result can be used in robotic bipedal walking.

B. Relationship with Previous Results

As noted above, Zeno behavior has long been studied in optimal control and nonsmooth mechanics. In 1960, Fuller showed that for certain constrained optimal control problems, the optimal controller makes an infinite number of switches in a finite amount of time [6]. Since then, this phenomenon

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has been observed and exploited in several optimal control problems [7]. In mechanics, Zeno behavior arising from impacts at the transition from bouncing to sliding is commonly observed [8]. While Zeno behavior can stall hybrid system simulations, time-stepping schemes for numerically integrating mechanical systems circumvent these problems because they do not require that impact times and locations be explicitly calculated [9]. Sufficient conditions for Zeno behavior, similar to the conditions for mechanical systems derived in this paper are given in [10], [11]. Conditions to rule out Zeno behavior have also been given for linear complementarity systems [12], [13], which are special hybrid models defined to capture discontinuous effects from nonsmooth mechanics, optimal control, and electrical circuits with diodes.

For hybrid systems, as studied in this paper, the theory of Zeno behavior has steadily matured with increased study of the dynamical aspects of hybrid systems. Hybrid automata, the precursors of the systems of this paper, were originally introduced to reason about embedded computing systems. Since a computer can only execute a finite number of operations in a finite amount of time, Zeno behavior was not allowed in early definitions of executions of hybrid automata [14], [15], [16]. As the scope of hybrid automaton research was extended to systems with rich continuous dynamics, attention to Zeno behavior increased to reason about well known examples such as the bouncing ball and Fuller's phenomenon. Early results focused on ruling out Zeno behavior using structural conditions [17], [18], or proving its existence using closed form solutions to simple differential equations [19], [20].

With increasing development of the qualitative, geometric theory hybrid systems [21], [22], connections between Zeno behavior and stability were recognized [23]. In [2], we gave Lyapunov-like sufficient conditions for asymptotic Zeno stability of isolated Zeno equilibria in a class of systems similar to that studied in this paper. This work was generalized in two separate directions in [24] and [4]. As mentioned above, Lagrangian hybrid systems with isolated Zeno equilibria must have one-dimensional configuration manifolds. The aim of [24], on which the current paper is based, was to extend the Lyapunov-like theory of Zeno stability to cover more complex examples, especially from mechanics, which typically have non-isolated Zeno equilibria. The results on Lagrangian hybrid systems from [24] were subsequently extended and refined in [25], [26], [27], [28], [29]. On the other hand, work in [4] exploited the connections between Zeno behavior and finite-time stability to give Lyapunov theorems (and associated converse theorems) for asymptotic Zeno stability general class hybrid systems which encompasses the models considered in [2], [24], and the current paper. Note, however, that because their work deals with asymptotic stability, it also cannot apply to nontrivial mechanical systems.

Similar to linearization in classical stability theory, Zeno stability can also be studied with local approximations [1], [3], [5]. In particular, the connection between Zeno behavior and homogeneity from [3], [5] was implicit in the early work of Fuller [6] and more fully explored subsequent work on optimal control [7] and relay systems [30].

II. Hybrid Systems & Zeno Behavior

In this section, we introduce the basic terminology used throughout paper. That is, we define hybrid systems, executions, and Zeno behavior. We study a restricted class of hybrid automata that strips away the nondeterminism and complicated graph structures allowing us to focus on consequences of the continuous dynamics.

Definition 1: A *hybrid system on a cycle* is a tuple:

$$\mathscr{H} = (\Gamma, D, G, R, F),$$

where

- $\Gamma = (Q, E)$ is a directed cycle, with
$$\begin{aligned} Q &= \{q_0, \ldots, q_{k-1}\}, \\ E &= \{e_0 = (q_0, q_1), e_1 = (q_1, q_2), \\ &\qquad \ldots, e_{k-1} = (q_{k-1}, q_0)\}. \end{aligned}$$
We denote the source of an edge $e \in E$ by $\mathrm{source}(e)$ and the target of an edge by $\mathrm{target}(e)$.
- $D = \{D_q\}_{q \in Q}$ is a set of *continuous domains*, where D_q is a smooth manifold.
- $G = \{G_e\}_{e \in E}$ is a set of *guards*, where $G_e \subseteq D_{\mathrm{source}(e)}$ is an embedded submanifold of $D_{\mathrm{source}(e)}$.
- $R = \{R_e\}_{e \in E}$ is a set of *reset maps*, where $R_e : G_e \subseteq D_{\mathrm{source}(e)} \to D_{\mathrm{target}(e)}$ is a smooth map.
- $F = \{f_q\}_{q \in Q}$, where $f_q : D_q \to TD_q$ is a Lipschitz vector field on D_q.

Remark 1: Note that if a hybrid system over a finite graph displays Zeno behavior, the graph must contain a cycle (see [17] and [18]). Therefore, beginning with hybrid systems defined on cycles greatly simplifies our analysis, while still capturing characteristic types of Zeno behavior.

Definition 2: An *execution* (or *solution*) of a hybrid system $\mathscr{H} = (\Gamma, D, G, R, F)$ is a tuple:

$$\chi = (\Lambda, I, \rho, C)$$

where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is a finite or infinite indexing set,
- $I = \{I_i\}_{i \in \Lambda}$ where for each $i \in \Lambda$, I_i is defined as follows: $I_i = [\tau_i, \tau_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \tau_N)$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, N finite. Here, for all $i, i+1 \in \Lambda$, $\tau_i \leq \tau_{i+1}$ with $\tau_i, \tau_{i+1} \in \mathbb{R}$, and $\tau_{N-1} \leq \tau_N$ with $\tau_{N-1}, \tau_N \in \mathbb{R}$. We set $\tau_0 = 0$ for notational simplicity.
- $\rho : \Lambda \to Q$ is a map such that for all $i, i+1 \in \Lambda$, $(\rho(i), \rho(i+1)) \in E$. This is the *discrete component* of the execution.
- $C = \{c_i\}_{i \in \Lambda}$ is a set of *continuous trajectories*, and they must satisfy $\dot{c}_i(t) = f_{\rho(i)}(c_i(t))$ for $t \in I_i$.

We require that when i, $i+1 \in \Lambda$,

$$\begin{array}{ll} \text{(i)} & c_i(t) \in D_{\rho(i)} \ \forall \ t \in I_i \\ \text{(ii)} & c_i(\tau_{i+1}) \in G_{(\rho(i), \rho(i+1))} \\ \text{(iii)} & R_{(\rho(i), \rho(i+1))}(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1}). \end{array} \tag{1}$$

When $i = |\Lambda| - 1$, we still require that (i) holds.

We call $c_0(0) \in D_{\rho(0)}$ the *continuous initial condition* and of χ. Likewise $\rho(0)$ is the discrete initial condition of χ.

Remark 2: To ensure that executions are deterministic, it is assumed that when an execution reaches a guard, the transition must be taken. Furthermore, to ensure that executions can be defined as t or i approach ∞, it is assumed that solutions to $\dot{x} = f_q(x)$ cannot leave D_q except through an associated guard, G_e.

This paper studies Zeno executions, defined as follows:

Definition 3: An execution χ is *Zeno* if $\Lambda = \mathbb{N}$ and

$$\lim_{i\to\infty} \tau_i = \sum_{i=0}^{\infty} \tau_{i+1} - \tau_i = \tau_\infty < \infty.$$

Here τ_∞ is called the *Zeno time*.

Zeno behavior displays strong connections with Lyapunov stability [2], [4]. Just as classical stability focuses on equilibria, much of the interesting Zeno behavior occurs near a special type of invariant set, termed Zeno equilibria.

Definition 4: A *Zeno equilibrium* of a hybrid system $\mathscr{H} = (\Gamma, D, G, R, F)$ is a set $z = \{z_q\}_{q\in Q}$ satisfying the following conditions for all $q \in Q$:

- For the unique edge $e = (q, q') \in E$
  - $z_q \in G_e$,
  - $R_e(z_q) = z_{q'}$,
- $f_q(z_q) \neq 0$.

A Zeno equilibrium $z = \{z_q\}_{q\in Q}$ is *isolated* if there is a collection of open sets $\{W_q\}_{q\in Q}$ such that $z_q \in W_q \subset D_q$, and $\{W_q\}_{q\in Q}$ contains no Zeno equilibria other than z. Otherwise, z is *non-isolated*.

Note that, in particular, the conditions given in Definition 4 imply that for all $i \in \{0, \ldots, k-1\}$,

$$R_{e_{i-1}} \circ \cdots \circ R_{e_0} \circ R_{e_{k-1}} \circ \cdots \circ R_{e_i}(z_i) = z_i.$$

That is, the element z_i is a fixed point under the reset maps composed in a cyclic manner. Furthermore, the assumptions from Remark 2 imply that any infinite execution with initial condition $c_0(0) \in z$ must be *instantaneously* Zeno (that is, $\tau_i = 0$ for all $i \in \mathbb{N}$).

Definition 4 captures notions that appear to be necessary for the Zeno phenomena studied in this paper. Indeed, unless the domains have geometric pathologies such as cusps or the vector fields are not locally Lipschitz, convergent, non-chattering Zeno executions (those with $\tau_i < \tau_{i+1}$ for infinitely many i) must converge to a Zeno equilibrium; see [21] Proposition 4.4. See [21] and [4] for examples of Zeno hybrid systems defined on cusps which do not have Zeno equilibria.

Finally, we give definitions that connect Zeno behavior to Lyapunov stability.

Definition 5: An execution $\chi = (\Lambda, I, \rho, C)$ is *maximal* if for all executions $\hat{\chi} = (\hat{\Lambda}, \hat{I}, \hat{\rho}, \hat{C})$ such that

$$\Lambda \subset \hat{\Lambda}, \qquad \bigcup_{j\in\Lambda} I_j \subset \bigcup_{j\in\hat{\Lambda}} \hat{I}_j,$$

and $c_j(t) = \hat{c}_j(t)$ for all $j \in \Lambda$ and $t \in I_j$, it follows that $\hat{\chi} = \chi$.

Definition 6: A Zeno equilibrium $z = \{z_q\}_{q\in Q}$ of a hybrid system $\mathscr{H} = (\Gamma, D, G, R, F)$ is:

- *bounded-time Zeno stable* if for every collection of open sets $\{U_q\}_{q\in Q}$ with $z_q \in U_q \subset D_q$ and every $\varepsilon > 0$, there is another collection of open sets $\{W_q\}_{q\in Q}$ with $z_q \in W_q \subset U_q$ such that if χ is a maximal execution with $c_0(0) \in W_{\rho(0)}$, then χ is Zeno with $\tau_\infty < \varepsilon$ and $c_i(t) \in U_{\rho(i)}$ for all $i \in \mathbb{N}$ and all $t \in I$.
- *bounded-time asymptotically Zeno stable* if it is bounded-time Zeno stable and there is a collection of open sets $\{W_q\}_{q\in Q}$ such that $z_q \in W_q \subset D_q$ and every Zeno execution $\chi = (\Lambda, I, \rho, C)$ with $c_0(0) \in W_{\rho(0)}$ converges to z as $i \to \infty$. More precisely, for any collection of open sets $\{U_q\}_{q\in Q}$ with $z_q \in U_q \subset D_q$, there is $N \in \mathbb{N}$ such that if $i \geq N$, then $c_i(t) \in U_{\rho(i)}$ for all $t \in I_i$.
- *bounded-time non-asymptotically Zeno stable* if it is bounded-time Zeno stable but not bounded-time asymptotically Zeno stable.

The following structural fact shows that isolatedness of a Zeno equilibrium dictates the type of Zeno stability properties it can display. While the theorem is independent of the main results of the paper, it clarifies the existing sufficient conditions for Zeno stability and adds context to our current work.

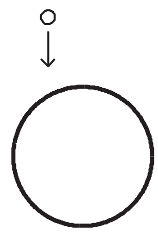
Theorem 1: *Let $z = \{z_q\}_{q\in Q}$ be a bounded-time Zeno stable equilibrium. Then z is bounded-time asymptotically Zeno stable if and only if z is isolated.*

Note the sharp contrast between Theorem 1 and classical stability theory. The standard theory of continuous dynamical systems focuses primarily on isolated equilibria without much apparent conceptual loss. In Zeno hybrid systems, however, we must consider non-isolated Zeno equilibria just to describe the non-asymptotic analog of Lyapunov stability.

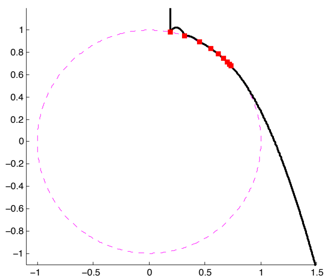
From the theorem, we learn that many of the recent sufficient conditions for Zeno stability have similar limitations, but for different reasons. The work in [3] and [4] requires bounded-time asymptotic Zeno stability (or the stronger global version), while [1] and [2] assume that the hybrid systems studied have isolated Zeno equilibria. None of the conditions in the papers listed above apply to the mechanical systems in this paper, since they have non-isolated Zeno equilibria.

Proof: Let z be an isolated Zeno equilibrium. By continuity, there is a collection of bounded neighborhoods $\{U_q\}_{q\in Q}$ containing no Zeno equilibria other than z, such that for all $q \in Q$, $z_q \in U_q \subset D_q$ and $f_q(x) \neq 0$ for all $x \in U_q$. From bounded-time local Zeno stability, there is another collection of neighborhoods $\{W_q\}_{q\in Q}$ such that all maximal executions with initial conditions in $\{W_q\}_{q\in Q}$ are all Zeno and never leave $\{U_q\}_{q\in Q}$. Let χ be any maximal execution such that $c_0(0) \in \{W_q\}_{q\in Q}$. Since χ is Zeno and bounded, Proposition 4.3 of [21] implies that there is a collection of points $\hat{z} = \{\hat{z}_q\}_{q\in Q}$ such that:

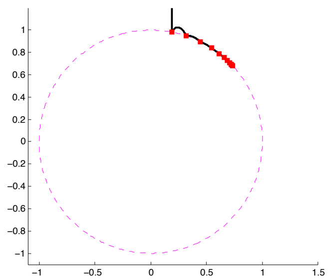
- $\hat{z}_q \in G_{(q,q')} \cap U_q$ for all $(q, q') \in E$,



(a)



(b)



(c)

Fig. 1. A ball moving through the plane under gravitational acceleration that bounces on a fixed circular surface. The simulations use $g = 1$, $e = 1/2$, starting from initial conditions of the form $c_0(0) = (x_1, x_2, 0, 0)$, with x_1 and x_2 varied. 1(b) When $(x_1, x_2) = (0.192, 1.2)$, the execution bounces several times before free falling to infinity. 1(c) Shifting x_1 to the left a small amount, so that $(x_1, x_2) = (0.191, 1.2)$, the execution becomes Zeno.

- $R_{(q,q')}(\hat{z}_q) = \hat{z}_{q'}$ for all $(q, q') \in E$,
- $c_i(t) \to \hat{z}_{\rho(i)}$ as $i \to \infty$.

The convergence, $c_i(t) \to \hat{z}_{\rho(i)}$, should be understood as follows. For each domain, let V_q be an arbitrary neighborhood of z_q. There exists N such that $i \geq N$ implies that $c_i(t) \in V_{\rho(i)}$ for all $t \in I_i$. Since $\hat{z}_q \in U_q$, it follows that $f_q(\hat{z}_q) \neq 0$ for all $q \in Q$. Therefore $\hat{z}$ is a Zeno equilibrium contained in $\{U_q\}_{q \in Q}$. By construction of U_q, we find that $\hat{z} = z$, and thus χ converges to z. We conclude that z is bounded-time asymptotically Zeno stable.

Conversely, let z be a non-isolated Zeno equilibrium. Then for any collection of neighborhood $\{U_q\}_{q \in Q}$, there is a Zeno equilibrium $\hat{z} = \{\hat{z}_q\}_{q \in Q}$ with $\hat{z} \neq z$ and $\hat{z}_q \in U_q$. Furthermore, any maximal execution with $c_0(0) = \hat{z}_{\rho(0)} \in U_{\rho(0)}$ is Zeno but does not converge to z. Therefore, z is not bounded-time asymptotically Zeno stable. ■

Example 1 (Bouncing Ball on a Circle): We illustrate the definitions and concepts above, and the theorems to follow by studying a ball bouncing on a circular surface (Figure 1(a)), modeled formally with the hybrid system

$$\mathscr{H}_{\mathbf{B}} = (\Gamma = (\{q\}, \{(q,q)\}, \{D_{\mathbf{B}}\}, \{G_{\mathbf{B}}\}, \{R_{\mathbf{B}}\}, \{f_{\mathbf{B}}\}),$$

where

$$\begin{array}{rcl} D_{\mathbf{B}} & = & \{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|x\| \geq 1\}, \\ G_{\mathbf{B}} & = & \{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|x\| = 1, \;\; x^T \dot{x} \leq 0\}, \end{array} \tag{2}$$

$$R_{\mathbf{B}}(x, \dot{x}) = \begin{pmatrix} x \\ \dot{x} - (1+e)(x^T \dot{x})x \end{pmatrix}, \quad f_{\mathbf{B}}(x, \dot{x}) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ 0 \\ -g \end{pmatrix}.$$

Here the reset map, $R_{\mathbf{B}}$, is a Newtonian impact model, with a coefficient of restitution, $0 \leq e \leq 1$, that describes an instantaneous jump in velocity when the ball impacts the circle. The vector field $f_{\mathbf{B}}$ models flight under gravitational acceleration.

Since $f_{\mathbf{B}}(x, \dot{x}) \neq 0$ on the entire continuous domain, the Zeno equilibria are exactly the fixed points of the reset map:

$$Z_{\mathbf{B}} = \{(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|x\| = 1, \;\text{ and }\; x^T \dot{x} = 0\}.$$

Note that $Z_{\mathbf{B}}$ is an infinite, connected set. Therefore, $\mathscr{H}_{\mathbf{B}}$ has no isolated Zeno equilibria. From Theorem 1, this bouncing ball system has no bounded-time asymptotically stable Zeno equilibria.

Turning to Zeno stability, the theory developed in this paper predicts that whenever $0 < e < 1$ and $(x^*, \dot{x}^*) \in \mathbb{R}^2 \times \mathbb{R}^2$ satisfies the following algebraic conditions:

$$\|x^*\| = 1, \qquad {x^*}^T \dot{x}^* = 0, \qquad \|\dot{x}^*\|^2 < g x_2^*, \tag{3}$$

the singleton set $\{(x^*, \dot{x}^*)\}$ is a bounded-time non-asymptotically Zeno stable Zeno equilibrium. Note how the conditions guarantee a non-compact continuum of bounded-time Zeno stable sets along the entire open upper half circle, even at points with nearly vertical tangent spaces. Furthermore, the theory developed in this paper and its extensions such as [28] can be used to numerically distinguish between Zeno executions and executions that take several bounces before free fall (Figures 1(b) and 1(c)).

III. Lyapunov Theory for Zeno Stability

This section presents Lyapunov-like sufficient conditions for Zeno stability. First, classical Lyapunov functions are interpreted as mapping solutions of differential equations to solutions of a one-dimensional differential inclusion, referred to as the target system. It is noted that by varying the target system, different specifications can be proved using Lyapunov-like techniques. Next, a class of target systems for proving Zeno stability are defined and studied. Finally, sufficient conditions for Zeno stability, based on Lyapunov-like reductions to the target systems are presented.

A. Classical Lyapunov Theory Reinterpretted

This subsection discusses classical Lyapunov theory, with empahasis on which components of the theory can be varied to reason about specifications other than classical stability. In standard treatments of Lyapunov theory, there is a differential equation

$$\dot{x} = f(x)$$

with $f : D \to \mathbb{R}^n$, and $f(0) = 0$. Here $D \subset \mathbb{R}^n$ is an open set with 0 in its interior. A continuously differentiable function $V : D \to \mathbb{R}$ is called a Lyapunov function if $V(0) = 0$ and for all nonzero $x \in D$

$$V(x) > 0 \quad \text{and} \quad \frac{\partial V}{\partial x}(x) f(x) \leq 0.$$

Assume that $x(t)$ is a solution to the differential equation. In standard presentations of Lyapunov theory, such as [31], the behavior of $V(x(t))$ merits limited discussion, due to its simplicity. To modify Lyapunov theory, however, the behavior of the image $V(x(t))$ is crucial and will thus be emphasized. First note that $V(x(t))$ satisfies the following differential inclusion on the non-negative real line:

$$\dot{v} \in \left\{ \begin{array}{ll} (-\infty, 0] & \text{for } v > 0 \\ \{0\} & \text{for } v = 0. \end{array} \right. \tag{4}$$

While $V(x(t))$ satisfies a differential inclusion, it typically does not satisfy a differential equation because V and its Lie derivative are typically not 1-1. In particular, it is often the case that there are two vectors x and $\hat{x}$ such that

$$V(x) = V(\hat{x}) \quad \text{but} \quad \frac{\partial V}{\partial x}(x) f(x) \neq \frac{\partial V}{\partial x}(\hat{x}) f(\hat{x}).$$

The target system, defined by equation (4), encodes simple stable dynamics that are flexible enough to describe convergence properties of the orignal dynamical system, through the use of a Lyapunov function. In a sense, the Lyapunov function "reduces" the stability properties of the original system to the stability properties of the target system. For Zeno stability, an analogous development holds. A simple class of hybrid systems is defined to serve as target systems. Then, Lyapunov-like functions are given to reduce Zeno stability properties of a given hybrid system to the stability properties of the associated target system.

B. *First Quadrant Interval Hybrid Systems*

This subsection introduces first quadrant interval hybrid systems, which serve as the targets for Lyapunov-like reductions for Zeno stability. First quadrant interval hybrid systems are a variant of first quadrant hybrid systems studied in [1] and [20]. We use the term "interval" since both the vector fields and reset maps are interval valued. See [32] and [33] for more on set valued functions and differential inclusions.

Definition 7: A *first quadrant interval* (**FQI**) *hybrid system* is a tuple

$$\mathscr{H}_{FQI} = (\Gamma, D, G, R, F)$$

where

- $\Gamma = (Q, E)$ is a directed cycle as in Definition 1.
- $D = \{D_q\}_{q \in Q}$ where for all $q \in Q$,
$$D_q = \mathbb{R}^2_{\geq 0} = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0,\ \ x_2 \geq 0\}.$$
- $G = \{G_e\}_{q \in Q}$ where for all $e \in E$,
$$G_e = \{(x_1, x_2)^T \in \mathbb{R}^2_{\geq 0} : x_1 = 0,\ \ x_2 \geq 0\}.$$
- $R = \{R_e\}_{e \in E}$ where for all $e \in E$, R_e is a set valued function defined by
$$\begin{aligned} R_e(0, x_2) \quad = \quad & \{(y_1, y_2)^T \in D_{q'} : y_1 = 0, \\ & y_2 \in [\gamma^l_e x_2, \gamma^u_e x_2]\}, \end{aligned}$$
for $\gamma^u_e \geq \gamma^l_e > 0$ and for all $(0, x_2)^T \in G_e$.
- $F = \{f_q\}_{q \in Q}$ where for all $q \in Q$, f_q is the (constant) set-valued function defined by
$$f_q(x) = \{(y_1, y_2)^T \in \mathbb{R}^2 : y_1 \in [\alpha^l_q, \alpha^u_q], y_2 \in [\beta^l_q, \beta^u_q]\}.$$

Definition 8: An *execution* of a first quadrant interval system, $\mathscr{H}_{FQI}$ is a tuple $\chi_{FQI} = (\Lambda, I, \rho, C)$ where

- Λ, I and ρ are defined as in Definition 2.
- $C = \{c_i\}_{i \in \Lambda}$ is a set of *continuous trajectories* that satisfy the differential inclusion $\dot{c}_i(t) \in f_{\rho(i)}(c_i(t))$ for $t \in I_i$.

We require that when i, $i+1 \in \Lambda$,

$$\begin{array}{ll} \text{(i)} & c_i(t) \in D_{\rho(i)}\ \forall\ t \in I_i \\ \text{(ii)} & c_i(\tau_{i+1}) \in G_{(\rho(i), \rho(i+1))} \\ \text{(iii)} & c_{i+1}(\tau_{i+1}) \in R_{(\rho(i), \rho(i+1))}(c_i(\tau_{i+1})). \end{array} \tag{5}$$

When $i = |\Lambda| - 1$, we still require that (i) holds.

As mentioned above, the class of first quadrant interval hybrid systems is motivated by their simple Zeno stability theory. Indeed, they are among the simplest systems that can demonstrate the non-chattering Zeno behavior of interest for this paper. The following theorem gives sufficient conditions for Zeno stability of first quadrant interval hybrid systems.

Theorem 2: *Let $\mathscr{H}_{FQI} = (\Gamma, D, G, R, F)$ be a first quadrant interval hybrid system. If $\alpha^u_q < 0 < \beta^l_q$ for all $q \in Q$, $\gamma^l_e > 0$ for all $e \in E$ and*

$$\prod_{i=0}^{|Q|-1} \left| \gamma^u_{e_i} \frac{\beta^u_{q_i}}{\alpha^u_{q_i}} \right| < 1$$

then the origin $\{0_q\}_{q \in Q}$ is bounded-time asymptotically Zeno stable.

Proof: Define $0 < \zeta < 1$ by

$$\zeta := \prod_{i=0}^{|Q|-1} \left| \gamma^u_{e_i} \frac{\beta^u_{q_i}}{\alpha^u_{q_i}} \right|. \tag{6}$$

Let χ_{FQI} be an execution of $\mathscr{H}_{FQI}$. Without loss of generality, assume that $c_0(0) \in D_{q_0}$. Since $f_q(x)_2 \geq \beta^l_q > 0$, the continuous trajectories travel upwards, away from the x_1-axis. Likewise, $f_q(x)_1 \leq \alpha^u_q < 0$ implies that the continuous trajectories travel left, towards the x_2-axis. Therefore, by construction, events are always guaranteed to occur and we can assume that $\Lambda = \mathbb{N}$. For simplicity, assume that $c_0(0)_2 = 0$. Dropping this assumption changes little, though the proofs become messier.

The hypothesis $\alpha^u_{\rho(i)} < 0$ implies that $c_i(t)_1 \leq c_i(\tau_i)_1 + \alpha^u_{\rho(i)}(t - \tau_i)$, and therefore

$$\tau_{i+1} - \tau_i \leq \left| \frac{c_i(\tau_i)_1}{\alpha^u_{\rho(i)}} \right|, \tag{7}$$

for all $i \geq 0$. Given the assumption that $c_0(0)_2 = 0$ and the fact that $c_i(\tau_i)_2 = 0$ for all $i \geq 1$, the continuous state at events must satisfy

$$c_i(\tau_{i+1})_2 \leq \beta^u_{\rho(i)}(\tau_{i+1} - \tau_i) \leq c_i(\tau_i)_1 \left| \frac{\beta^u_{\rho(i)}}{\alpha^u_{\rho(i)}} \right|. \tag{8}$$

Thus, after the events the continuous state satisfies

$$\begin{aligned} c_{i+1}(\tau_{i+1})_1 \quad & \leq \quad \gamma^u_{(\rho(i), \rho(i+1))} c_i(\tau_{i+1})_2 \\ & \leq \quad c_i(\tau_i)_1 \gamma^u_{(\rho(i), \rho(i+1))} \left| \frac{\beta^u_{\rho(i)}}{\alpha^u_{\rho(u)}} \right|. \end{aligned} \tag{9}$$

Inductively combining the bounds from equation (9) gives a bound in terms of $c_0(0)_1$:

$$c_i(\tau_i)_1 \leq c_0(0)_1 \prod_{j=0}^{i-1} \left| \gamma^u_{(\rho(j),\rho(j+1))} \frac{\beta^u_{\rho(j)}}{\alpha^u_{\rho(j)}} \right|, \qquad (10)$$

for all $i \geq 0$.

To prove stability and asymptotic convergence note that $\alpha^u_q < 0 < \beta^l_q$ implies that $c_i(t)_1 \leq c_i(\tau_i)_1$ and $c_i(t)_2 \leq c_i(\tau_{i+1})_2$ for all $t \in I_i$. Combining equations (8), (9), and (10) gives the bound

$$\begin{aligned} &\|c_i(t)\| \\ &\leq \quad c_i(\tau_i)_1 + c_i(\tau_{i+1})_2 \\ &\leq \quad \left(1 + \left| \frac{\beta^u_{\rho(i)}}{\alpha^u_{\rho(i)}} \right| \right) c_i(\tau_i)_1 \\ &\leq \quad \left(1 + \left| \frac{\beta^u_{\rho(i)}}{\alpha^u_{\rho(i)}} \right| \right) c_0(0)_1 \prod_{j=0}^{i-1} \left| \gamma^u_{(\rho(j),\rho(j+1))} \frac{\beta^u_{\rho(j)}}{\alpha^u_{\rho(j)}} \right| \end{aligned}$$

Since the product in the last inequality converges to 0 as $i \to \infty$, executions with $c_0(0)_1$ small must remain near the origin, and $c_i(t) \to 0_{\rho(i)}$ as $i \to \infty$.

Combining equations (6), (7) and (10) and proves that χ is Zeno:

$$\begin{aligned} &\sum_{i=0}^{\infty} \tau_{i+1} - \tau_i \\ &\leq \quad c_0(0)_1 \sum_{i=0}^{\infty} \frac{1}{|\alpha^u_{\rho(i)}|} \prod_{j=0}^{i-1} \left| \gamma^u_{(\rho(j),\rho(j+1))} \frac{\beta^u_{\rho(j)}}{\alpha^u_{\rho(j)}} \right| \\ &= \quad c_0(0)_1 \left(\sum_{j=0}^{|Q|-1} \frac{1}{|\alpha^u_{q_j}|} \prod_{k=0}^{j-1} \left| \gamma^u_{e_k} \frac{\beta^u_{q_k}}{\alpha^u_{q_k}} \right| \right) \cdot \left(\sum_{i=0}^{\infty} \zeta^i \right) \\ &< \quad \infty. \end{aligned}$$

Furthermore, note that the bound on the Zeno time goes to zero as $c_1(0)_1 \to 0$. ∎

Theorem 2 can also be proved using Lyapunov methods from [4] or the homogeneity methods from [5], but the specific form of convergence of $c_i(t)$ to the Zeno equilibrium exploited in the proof above is used to prove Theorem 3.

The Target Systems from [4]. It is worth comparing first quadrant interval hybrid systems with the class of targets used in [4]. These targets are defined for $v \in [0, \infty)$ by

$$\begin{aligned} \dot{v} \quad &\in \quad \begin{cases} (-\infty, -1] & \text{for } v > 0 \\ 0 & \text{for } v = 0 \end{cases} \\ v^+ \quad &\in \quad [0, v - \kappa(v)] \quad \text{for } v \geq 0, \end{aligned}$$

where κ is some class $\mathcal{K}_\infty$ function. In the framework of [4], solutions for this hybrid system flow according to the differential inclusion almost everywhere, but can make a jumps according to the difference inclusion at nondeterministic times. If any such solution starts at $v = v_0$, then it must converge to 0 in time at most v_0. Thus, these target systems exhibit finite-time convergence but the solutions may or may not be Zeno. To guarantee non-chattering Zeno behavior, extra conditions on the original hybrid system must be checked.

C. Sufficient Conditions for Zeno Stability Through Reduction to **FQI** *Hybrid Systems*

This subsection gives the main Lyapunov-type theorem of this paper. Our theorem uses special Lyapunov-like functions to map executions of complex hybrid systems down to executions of **FQI** hybrid systems, thus transferring some Zeno stability properties from Theorem 2. The theorem applies to both isolated and non-isolated Zeno equilibria. Therefore, by Theorem 1, our sufficient conditions can imply bounded-time asymptotic or non-asymptotic Zeno stability, depending on the type of Zeno equilibrium in question.

Reduction conditions. Let $z = \{z_q\}_{q \in Q}$ be a Zeno equilibrium (not necessarily isolated) of a hybrid system $\mathscr{H} = (\Gamma, D, G, R, F)$, $\{W_q\}_{q \in Q}$ be a collection of open sets with $z_q \in W_q \subseteq D_q$ and $\{\psi_q\}_{q \in Q}$ be a collection of C^1 maps; these are "Lyapunov-like" functions, with

$$\psi_q : W_q \subseteq D_q \to \mathbb{R}^2_{\geq 0}.$$

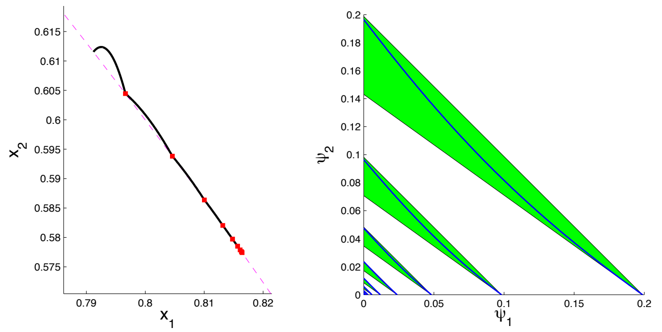
Consider the following conditions:

- **R**1: $\psi_q(z_q) = 0$ for all $q \in Q$.
- **R**2: If $(q, q') \in E$, then $\psi_q(x)_1 = 0$ if and only if $x \in G_{(q,q')} \cap W_q$.
- **R**3: $d\psi_q(z_q)_1 f_q(z_q) < 0 < d\psi_q(z_q)_2 f_q(z_q)$ for all $q \in Q$.
- **R**4: $\psi_{q'}(R_{(q,q')}(x))_2 = 0$ and there exist constants $0 < \gamma^l_e \leq \gamma^u_e$ such that
$$\psi_{q'}(R_{(q,q')}(x))_1 \in \left[\gamma^l_{(q,q')} \psi_q(x)_2, \gamma^u_{(q,q')} \psi_q(x)_2\right]$$
for all $x \in G_{(q,q')} \cap W_q$ and all $(q, q') \in E$.
- **R**5:
$$\prod_{i=0}^{|Q|-1} \left| \gamma^u_{e_i} \frac{d\psi_{q_i}(z_{q_i})_2 f_{q_i}(z_{q_i})}{d\psi_{q_i}(z_{q_i})_1 f_{q_i}(z_{q_i})} \right| < 1.$$
- **R**6: There exists $K \geq 0$ such that
$$\|R_{(q,q')}(x) - z_{q'}\| \leq \|x - z_q\| + K\psi_q(x)_2$$
for all $x \in G_{(q,q')} \cap W_q$ and all $(q, q') \in E$.

Remark 3: Just as the conditions on classical Lyapunov functions guarantee that solutions of a dynamical system can be mapped to solutions of a stable one-dimensional system, conditions **R**1-**R**5 guarantee that ψ_q can be used to map executions of $\mathscr{H}$ to executions of a Zeno stable **FQI** hybrid system. Condition **R**6 is used to guarantee that Zeno behavior occurs before the execution can leave the neighborhoods, W_q. Figure 2 depicts the reduction conditions applied to the example of a ball on a circle.

Theorem 3: *Let $\mathscr{H}$ be a hybrid system with a Zeno equilibrium $z = \{z_q\}_{q \in Q}$. If there exists a collection of open sets $\{W_q\}_{q \in Q}$ with $z_q \in W_q \subseteq D_q$ and maps $\{\psi_q\}_{q \in Q}$ satisfying conditions* **R***1-***R***6, then z is bounded-time Zeno stable.*

Before proving this theorem we state the following corollary, which follows from combining Theorems 1 and 3.



(a) (b)

Fig. 2. 2(a) A Zeno execution of the ball on a circle. 2(b) The execution is mapped to an execution of a first quadrant interval hybrid system. Since the continuous domain of the bouncing ball system is four-dimensional, the mapping takes executions of the bouncing ball system to a executions of a system with lower dimension.

Corollary 1: *Let $\mathscr{H}$ be a hybrid system with a Zeno equilibrium $z = \{z_q\}_{q\in Q}$ satisfying the conditions of Theorem 3. If z is an isolated Zeno equilibrium, then z is bounded-time asymptotically Zeno stable. Otherwise, if z is a non-isolated Zeno equilibrium, then z is bounded-time non-asymptotically Zeno stable.*

Constructing a FQI hybrid system. The proof of Theorem 3 is based on the Zeno behavior of a first quadrant interval system $\mathscr{H}_{\rm FQI}$ constructed from the reduction conditions as follows. Assume that $\mathscr{H}$ is a hybrid system satisfying **R**1-**R**5. Pick α_q^l, α_q^u, β_q^l and β_q^u such that

$$\alpha_q^l < d\psi_q(z_q)_1 f_q(z_q) < \alpha_q^u < 0 \beta_q^l < d\psi_q(z_q)_2 f_q(z_q) < \beta_q^u$$

for all $q \in Q$ and

$$\prod_{i=0}^{|Q|-1} \left| \gamma_{e_i}^u \frac{\beta_{q_i}^u}{\alpha_{q_i}^u} \right| < 1,$$

where $\gamma_{e_i}^u$ is given by **R**4. The constants α_q^l, α_q^u, β_q^l, β_q^u, $\gamma_{(q,q')}^l$ and $\gamma_{(q,q')}^u$ (with $\gamma_{(q,q')}^l$ also given by **R**4) thus define a first quadrant interval system $\mathscr{H}_{\rm FQI}$, on the same graph as $\mathscr{H}$, satisfying the conditions of Theorem 2 due to conditions **R**3-**R**5. Thus all executions of $\mathscr{H}_{\rm FQI}$ extend to Zeno executions.

Proof of Theorem 3: The results are local and we can work in coordinate charts. Thus, it can be assumed without loss of generality that $D_q \subset \mathbb{R}^{n_q}$, for some n_q, and that $z_q = 0$. The proof relies on the following three claims:

1) Let $\chi = (\Lambda, \rho, I, C)$ be an execution of $\mathscr{H}$. For all $\mu > 0$, there exist η and T with $0 < \eta < \mu$ and $T > 0$ such that $\|c_0(0)\| < \eta$ implies that $\|c_i(t)\| < \mu$ for all $t < T$ for which $c_i(t)$ is well defined.
2) Let $\chi = (\Lambda, \rho, I, C)$ be an execution of $\mathscr{H}$. There exists $\mu > 0$ such that if $\|c_i(t)\| < \mu$ for all $i \in \Lambda$ and all $t \in I_i$, then $\chi_{\rm FQI} = (\Lambda, \rho, I, \Psi \circ C)$, where $\Psi \circ C = \{\psi_{\rho(i)} \circ c_i\}_{i\in\Lambda}$, is an execution of of $\mathscr{H}_{\rm FQI}$.
3) Let $\hat{\chi} = (\hat{\Lambda}, \hat{\rho}, \hat{I}, \hat{C})$ be an execution of $\mathscr{H}_{\rm FQI}$ with $\hat{\Lambda} = \mathbb{N}$ and Zeno time $T_{\rm Zeno}$. For all $T > 0$ there exists $\delta > 0$ such that $\|\hat{c}_0(0)\| < \delta$ implies that $T_{\rm Zeno} < T$.

Before proving the claims, it will be shown how they imply the theorem. The claims imply that positive numbers η, μ, T, and δ can be chosen such that:

- $\|c_0(0)\| < \eta$ implies that $\|c_i(t)\| < \mu$ for all $t < T$ for which $c_i(t)$ is defined.
- $\|c_i(t)\| < \mu$ for all $i \in \Lambda$ and all $t \in I_i$ implies that $\chi_{\rm FQI}$ is an execution of $\mathscr{H}_{\rm FQI}$.
- $\|c_0(0)\| < \eta$ implies that $\|\psi_{\rho(0)} \circ c_0(0)\| < \delta$.
- $\|\hat{c}_0(0)\| < \delta$ implies that $T_{\rm Zeno} < T$.

Let $\tilde{\chi}$ be a maximal execution with $\|c_0(0)\| < \eta$. Assume for the sake of contradiction that $\tilde{\chi}$ is not Zeno. Let $\chi = (\Lambda, \rho, I, C)$ be the restriction of χ to times with $t < T$. Then, by choice of η, μ, and T, it follows that $\chi_{\rm FQI} = (\Lambda, \rho, I, \Psi \circ C)$ is an execution of $\mathscr{H}_{\rm FQI}$. By maximality and the assumption that $\tilde{\chi}$ is not Zeno, $\tilde{\chi}$ and thus χ must be defined for all t such that $T_{\rm Zeno} < t < T$. Therefore $\chi_{\rm FQI}$ must be defined for $t > T_{\rm Zeno}$, which is past the Zeno time, giving a contradiction. Therefore $\tilde{\chi} = \chi$ is Zeno with $\|c_i(t)\| < \mu$ for all i and t. It follows that $\{0_q\}_{q\in Q}$ is bounded-time Zeno stable.

Now the claims will be proved. First, note that Claim 3 follows from the proof of Theorem 2.

Next Claim 2 will be proved. By continuity, there exists $\mu > 0$ such that for all $q \in Q$ and for all $x \in W_q$ with $\|x\| < \mu$,

$$\alpha_q^l < d\psi_q(x)_1 f_q(x) < \alpha_q^u < 0 < \beta_q^l < d\psi_q(x)_2 f_q(x) < \beta_q^u,$$

wherein it follows that $\chi_{\rm FQI}$ satisfies the conditions of $\mathscr{H}_{\rm FQI}$ by construction. Indeed, $\psi_{\rho(i)}(c_i(t))$ satisfies the differential inclusion:

$$\frac{d}{dt}\psi_{\rho(i)}(c_i(t)) \in \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \in [\alpha_{\rho(i)}^l, \alpha_{\rho(i)}^u], \\ x_2 \in [\beta_{\rho(i)}^l, \beta_{\rho(i)}^u]\}.$$

Condition **R**2 guarantees that an event of $\chi_{\rm FQI}$ occurs if and only if an event occurs in χ. Condition **R**4 guarantees that $\chi_{\rm FQI}$ satisfies the first quadrant interval system reset condition defined by γ_e^l and γ_e^u. Therefore, $\chi_{\rm FQI}$ is an execution of $\mathscr{H}_{\rm FQI}$.

Finally, Claim 1 will be proved. Assume that $\|c_0(0)\| < \eta$ and $\|c_i(t)\| \geq \mu$. The goal is to choose η and $T > 0$ such that $t \geq T$ must hold. Let M be such that $\|f_q(x)\| \leq M$ for all x such that $\|x\| \leq \mu$. Then the total change in norm is bounded as

$$\|c_i(t)\| - \|c_0(0)\| \leq Mt + \sum_{j=0}^{i-1}(\|c_{j+1}(\tau_{j+1})\| - \|c_j(\tau_{j+1})\|),$$

where the first term is due to flow, while the second term is due to jumps. Applying condition **R**6 shows that $\|c_{j+1}(\tau_{j+1})\| - \|c_j(\tau_{j+1})\| \leq K\psi_{\rho(j)}(c_j(\tau_{j+1}))_2$, and therefore the change in norm can be bounded as

$$\|c_i(t)\| - \|c_0(0)\| \leq Mt + K\sum_{j=0}^{i-1}\psi_{\rho(j)}(c_j(\tau_{j+1}))_2.$$

Arguing as in the proof of Theorem 2 shows that

$$\psi_{\rho(j)}(c_j(\tau_{j+1}))_2 \leq \prod_{p=1}^{j} \left| \gamma^u_{\rho(p-1),\rho(p)} \frac{\beta^u_{\rho(p)}}{\alpha^u_{\rho(p)}} \right| \cdot \left(\psi_{\rho(0)}(c_0(0))_2 + \left| \frac{\beta^u_{\rho(0)}}{\alpha^u_{\rho(0)}} \right| \psi_{\rho(0)}(c_0(0))_1 \right),$$

with the product decreasing geometrically in j. By continuity of ψ_q, it follows that there is a continuous, increasing function $g(\eta)$ with $g(0) = 0$ such that

$$\sum_{j=0}^{i-1} \psi_{\rho(j)}(c_j(\tau_{j+1}))_2 \leq g(\eta)$$

whenever $\|c_0(0)\| < \eta$. Thus, the following chain of inequalities holds:

$$\mu - \eta \leq \|c_i(t)\| - \|c_0(0)\| \leq Mt + Kg(\eta).$$

Therefore, if η and T are chosen such that $\eta + Kg(\eta) < \mu$ and $T < \frac{\mu-\eta-Kg(\eta)}{M}$, it follows that $t > T$. ■

Example 2: The water tank system is a well-known early example of a Zeno hybrid automaton [18]. It is described as a hybrid system on a cycle by $\mathscr{H}_{\mathbf{W}} = ((\{q_0, q_1\}, \{e_0, e_1\}), D_{\mathbf{W}}, G_{\mathbf{W}}, R_{\mathbf{W}}, F_{\mathbf{W}})$, with $D_{\mathbf{W}} = \{D_{q_0}, D_{q_1}\}$, $G_{\mathbf{W}} = \{G_{e_0}, G_{e_1}\}$, $R_{\mathbf{W}} = \{R_{e_0}, R_{e_1}\}$, and $F_{\mathbf{W}} = \{F_{q_0}, F_{q_1}\}$ given by

$$\begin{aligned} D_{q_0} = D_{q_1} &= \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0\}, \\ G_{e_0} &= \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = 0,\ x_2 \geq 0\}, \\ G_{e_1} &= \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 = 0\}, \end{aligned}$$

$$R_{e_0}(x) = R_{e_1}(x) = x,$$

$$F_{q_0}(x) = \begin{pmatrix} -v_1 \\ w - v_2 \end{pmatrix}, \quad F_{q_1}(x) = \begin{pmatrix} w - v_1 \\ -v_2 \end{pmatrix}.$$

Here w, v_1, and v_2 are positive numbers with $v_1, v_2 < w$. Zeno behavior can be proved using Theorem 3 with functions

$$\psi_{q_0}((x_1, x_2)^T) = (x_1, x_2)^T, \quad \psi_{q_1}((x_1, x_2)^T) = (x_2, x_1)^T,$$

and constants $\gamma^l_{e_0} = \gamma^l_{e_1} = \gamma^u_{e_0} = \gamma^u_{e_1} = 1$, $K = 0$. Condition **R**5 reduces to $\frac{(w-v_1)(w-v_2)}{v_1 v_2} < 1$, which is equivalent to the condition for Zeno behavior from [18], $w < v_1 + v_2$.

IV. Application to Simple Hybrid Mechanical Systems

This section develops Zeno stability theory for a simple model of mechanical systems undergoing impacts, known as Lagrangian hybrid systems. We begin by introducing the basic definitions of Lagrangian hybrid systems (see [26], [27], [34] for more on systems of this form modeled with the framework of this paper and see [10], [11], [35], [36] for mechanics based formulations). We then specialize the main result of this paper, Theorem 3, to this class of systems to give easily verifiable sufficient conditions for Zeno behavior in Lagrangian hybrid systems. Finally, we conclude by summarizing an application to bipedal robots with knee-bounce.[1]

A. Lagrangian Hybrid Systems

Lagrangians. Consider a configuration space[2] Θ and a Lagrangian $L : T\Theta \to \mathbb{R}$ given in coordinates by:

$$L(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} - U(\theta) \tag{11}$$

where $M(\theta)$ is positive definite and symmetric and $U(\theta)$ is the potential energy. For the sake of simplicity, we assume $\Theta \subset \mathbb{R}^n$ since all our results are local, i.e., we can work within a coordinate chart. The equations of motion are then given in coordinates by the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0.$$

In the case of Lagrangians of the form given in (11), the vector field associated to L takes the familiar form

$$\dot{x} = f_L(x) = \begin{pmatrix} \dot{\theta} \\ M(\theta)^{-1}(-C(\theta, \dot{\theta})\dot{\theta} - N(\theta)) \end{pmatrix}. \tag{12}$$

where $x = (\theta^T, \dot{\theta}^T)^T$, $C(\theta, \dot{\theta})$ is the *Coriolis matrix* and $N(\theta) = \frac{\partial U}{\partial \theta}(\theta)$.

This process of associating a dynamical system to a Lagrangian will be mirrored in the setting of hybrid systems. First, we introduce the notion of a hybrid Lagrangian.

Definition 9: A *hybrid Lagrangian* is a tuple, $\mathbf{L} = (\Theta, L, h)$, where

- $\Theta \subset \mathbb{R}^n$ is the configuration space,
- $L : T\Theta \to \mathbb{R}$ is a Lagrangian of the form given in (11),
- $h : \Theta \to \mathbb{R}$ is a *unilateral constraint function*, where we assume that 0 is a regular value of h (to ensure that $h^{-1}(\{0\})$ is a smooth manifold).

To concretely illustrate the hybrid Lagrangian concepts of the rest of the section, we will return to the ball on a circle example discussed in Section II.

Example 3 (Ball on a Circle): Recall the ball bouncing on a circular surface introduced in Example 1 (c.f. Figure 1(a)). This system has a natural hybrid Lagrangian description: $\mathbf{B} = (\Theta_{\mathbf{B}}, L_{\mathbf{B}}, h_{\mathbf{B}})$, where $\Theta_{\mathbf{B}} = \mathbb{R}^2$, and for $x = (x_1, x_2)$,

$$L_{\mathbf{B}}(x, \dot{x}) = \frac{1}{2}m\|\dot{x}\|^2 - mgx_2, \qquad h_{\mathbf{B}}(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}.$$

We will show that the hybrid model of this system (as introduced in Example 1) can be constructed from the hybrid Lagrangian describing the system.

Domains from constraints. Given a smooth (unilateral constraint) function $h : \Theta \to \mathbb{R}$ on a configuration space Θ such that 0 is a regular value of h, we can construct a domain and a guard explicitly. Define the domain, D_h, as the manifold (with boundary):

$$D_h = \{(\theta, \dot{\theta}) \in T\Theta : h(\theta) \geq 0\}. \tag{13}$$

[1]While bipedal robots could be studied in other nonsmooth mechanics frameworks, throughout the literature on biped robots, they are typically modeled by hybrid systems (sometimes termed *systems with impulsive effects* [37], but still equivalent to the hybrid systems as defined in this paper [38]).

[2]Note that we denote the configuration space by Θ rather than Q, due to the fact that Q denotes the vertices of the graph of a hybrid system.

Similarly, we have an associated guard, G_h, defined as the following submanifold of D_h:

$$G_h = \{(\theta, \dot\theta) \in T\Theta : h(\theta) = 0 \text{ and } dh(\theta)\dot\theta \leq 0\}, \qquad (14)$$

where $dh(\theta) = \begin{pmatrix} \frac{\partial h}{\partial \theta_1}(\theta) & \cdots & \frac{\partial h}{\partial \theta_n}(\theta) \end{pmatrix}$. Note that 0 is a regular value of h if and only if $dh(\theta) \neq 0$ whenever $h(\theta) = 0$.

Lagrangian Hybrid Systems. Given a hybrid Lagrangian $\mathbf{L} = (\Theta, L, h)$, the *Lagrangian hybrid system associated to* $\mathbf{L}$ is the hybrid system

$$\mathscr{H}_{\mathbf{L}} = (\Gamma = (\{q\}, \{(q,q)\}), D_{\mathbf{L}}, G_{\mathbf{L}}, R_{\mathbf{L}}, F_{\mathbf{L}}),$$

where $D_{\mathbf{L}} = \{D_h\}$, $F_{\mathbf{L}} = \{f_L\}$, $G_{\mathbf{L}} = \{G_h\}$ and $R_{\mathbf{L}} = \{R_h\}$ with the reset map given by the Newtonian impact equation $R_h(\theta, \dot\theta) = (\theta, P(\theta, \dot\theta))$, with

$$P(\theta, \dot\theta) = \dot\theta - (1+e)\frac{dh(\theta)\dot\theta}{dh(\theta)M(\theta)^{-1}dh(\theta)^T} M(\theta)^{-1} dh(\theta)^T. \qquad (15)$$

Here $0 \leq e \leq 1$ is the coefficient of restitution.

Example 4 (Ball on a Circle): From the hybrid Lagrangian $\mathbf{B} = (\Theta_{\mathbf{B}}, L_{\mathbf{B}}, h_{\mathbf{B}})$ we obtain the Lagrangian hybrid system associated to $\mathbf{B}$:

$$\mathscr{H}_{\mathbf{B}} = (\Gamma = (\{q\}, \{(q,q)\}), D_{\mathbf{B}}, G_{\mathbf{B}}, R_{\mathbf{B}}, F_{\mathbf{B}})$$

It can be checked that this hybrid model is the model $\mathscr{H}_{\mathbf{B}}$ introduced in Example 1, i.e., $D_{\mathbf{B}}$, $G_{\mathbf{B}}$, $R_{\mathbf{B}}$, $F_{\mathbf{B}}$ as given in (2) are obtained through (13), (14), (15) and (12), respectively.

B. Sufficient Conditions for Zeno Behavior in Lagrangian Hybrid Systems

This subsection presents sufficient conditions for bounded-time Zeno stability of Lagrangian hybrid systems, based on an explicitly constructed Lyapunov-like function. The paper [10] proves a special case of the main result in this section, Theorem 4, for a class of Lagrangian hybrid systems with configuration manifolds of dimension two. If the potential energy is a convex function and the domain specified by the unilateral constraint is a convex set, global Zeno stability results have been proved in [11]. Of course, the convexity assumptions preclude local phenomena, in which some executions are Zeno while others are not, occurring in the bouncing ball example above or knee-bounce example below.

First, however, we examine the Zeno equilibria of Lagrangian hybrid systems, observing that isolated Zeno equilibria only occur in systems with one-dimensional configuration manifolds. Thus, no Lagrangian hybrid system with configuration manifold of dimension greater than one can have bounded-time asymptotically stable Zeno equilibria.

Zeno equilibria in Lagrangian hybrid systems. If $\mathscr{H}_{\mathbf{L}}$ is a Lagrangian hybrid system, then applying the definition of Zeno equilibria and examining the special form of the reset maps shows that $z = \{(\theta^*, \dot\theta^*)\}$ is a Zeno equilibrium if and only if

$$f_{\mathbf{L}}(\theta^*, \dot\theta^*) \neq 0, \quad h(\theta^*) = 0,$$
$$dh(\theta^*)\dot\theta^* \leq 0, \quad \dot\theta^* = P(\theta^*, \dot\theta^*).$$

Furthermore, the form of P implies that $\dot\theta^* = P(\theta^*, \dot\theta^*)$ holds if and only if $dh(\theta^*)\dot\theta^* = 0$. Therefore the set of all Zeno equilibria for a Lagrangian hybrid system is given by the surfaces in $T\Theta$:

$$Z_h = \{(\theta, \dot\theta) \in T\Theta : f_{\mathbf{L}}(\theta, \dot\theta) \neq 0,\ h(\theta) = 0,\ dh(\theta)\dot\theta = 0\}.$$

Note that if $\dim(\Theta) > 1$, the Lagrangian hybrid system has no isolated Zeno equilibria.

Theorem 4: *Let $\mathscr{H}_{\mathbf{L}}$ be a Lagrangian hybrid system and $(\theta^*, \dot\theta^*) \in D_h$. If the coefficient of restitution satisfies $0 < e < 1$ and $(\theta^*, \dot\theta^*)$ satisfies*

$$h(\theta^*) = 0, \quad \dot h(\theta^*, \dot\theta^*) = 0, \quad \ddot h(\theta^*, \dot\theta^*) < 0,$$

then $\{(\theta^*, \dot\theta^*)\}$ is a bounded-time stable Zeno equilibrium.

Here $\dot h(\theta^*, \dot\theta^*) = dh(\theta^*)\dot\theta^*$ and

$$\begin{aligned}\ddot h(\theta^*, \dot\theta^*) \;=\;& (\dot\theta^*)^T H(h(\theta^*))\dot\theta^* + \\ & dh(\theta^*)M(\theta^*)^{-1}(-C(\theta^*, \dot\theta^*)\dot\theta^* - N(\theta^*)),\end{aligned}$$

where $H(h(\theta^*))$ is the Hessian of h at θ^*.

Proof: First note that $\{(\theta^*, \dot\theta^*)\}$ is a Zeno equilibrium. Indeed, $d\dot h(\theta^*, \dot\theta^*) f_{\mathbf{L}}(\theta^*, \dot\theta^*) = \ddot h(\theta^*, \dot\theta^*) \neq 0$ implies that $f_{\mathbf{L}}(\theta^*, \dot\theta^*) \neq 0$. Then the conditions $h(\theta^*) = 0$ and $\dot h(\theta^*, \dot\theta^*) = 0$ imply that $\{(\theta^*, \dot\theta^*)\}$ is a Zeno equilibrium.

Let V be a small neighborhood of $(\theta^*, \dot\theta^*)$ and assume (by passing to a coordinate chart) that $V \subset \mathbb{R}^{2n}$ with Euclidean norm. Let K satisfy

$$K > \frac{1+e}{2}\frac{\|M(\theta^*)^{-1}dh(\theta^*)^T\|}{dh(\theta^*)M(\theta^*)^{-1}dh(\theta^*)^T}.$$

We verify that the constants $\gamma_h^u = \gamma_h^l = e$, K and the function

$$\psi_h(\theta, \dot\theta) = \begin{pmatrix} \dot h(\theta, \dot\theta) + \sqrt{\dot h(\theta, \dot\theta)^2 + 2h(\theta)} \\ -\dot h(\theta, \dot\theta) + \sqrt{\dot h(\theta, \dot\theta)^2 + 2h(\theta)} \end{pmatrix} \qquad (16)$$

satisfy conditions **R**1-**R**6 on V.

R1: Since $(\theta^*, \dot\theta^*)$ is a Zeno equilibrium, $h(\theta^*) = 0$ and $\dot h(\theta^*, \dot\theta^*) = dh(\theta^*)\dot\theta^* = 0$. Thus $\psi_h(\theta^*, \dot\theta^*) = 0$.

R2: Since $h(\theta) \geq 0$, $\psi_h(\theta, \dot\theta)_1 = 0$ if and only if $h(\theta) = 0$ and $\dot h(\theta, \dot\theta) \leq 0$. So $\psi_h(\theta, \dot\theta)_1 = 0$ if and only if $(\theta, \dot\theta) \in G_h$.

R3: The square root in the definition of ψ_h creates some differentiability problems at the Zeno equilibrium.

Assume V is small enough that V contains no equilibria of $f_{\mathbf{L}}$. Then $(D_h \setminus Z_h) \cap V$ has the form

$$(D_h \setminus Z_h) \cap V = \{(\theta, \dot\theta) \in V : h(\theta) > 0 \text{ or } \dot h(\theta, \dot\theta) \neq 0\},$$

and that ψ_h is continuously differentiable on $(D_h \setminus Z_h) \cap V$ with Lie derivative given by

$$\dot\psi_h(\theta, \dot\theta) = \begin{pmatrix} \ddot h(\theta, \dot\theta) + \frac{\dot h(\theta, \dot\theta)}{\sqrt{\dot h(\theta, \dot\theta)^2 + 2h(\theta)}}\left(\ddot h(\theta, \dot\theta) + 1\right) \\ -\ddot h(\theta, \dot\theta) + \frac{\dot h(\theta, \dot\theta)}{\sqrt{\dot h(\theta, \dot\theta)^2 + 2h(\theta)}}\left(\ddot h(\theta, \dot\theta) + 1\right) \end{pmatrix}. \qquad (17)$$

Recall that $\ddot h(\theta^*, \dot\theta^*) < 0$. It follows from the definitions that scaling h by a positive constant does not change D_h, G_h or R_h. Therefore we can assume that $\ddot h(\theta^*, \dot\theta^*) = -1$.

While the function $\frac{\dot{h}(\theta,\dot{\theta})}{\sqrt{\dot{h}(\theta,\dot{\theta})^2+2h(\theta)}}$ may not have a unique limit as $(\theta,\dot{\theta}) \to (\theta^*,\dot{\theta}^*)$ it remains bounded on $(D_h \backslash Z_h) \cap V$:

$$\frac{|\dot{h}(\theta,\dot{\theta})|}{\sqrt{\dot{h}(\theta,\dot{\theta})^2+2h(\theta)}} \leq 1.$$

Therefore, $\dot{\psi}_h$ has the well defined limit

$$\lim_{(\theta,\dot{\theta})\in(D_h\backslash Z_h)\cap V,\ (\theta,\dot{\theta})\to(\theta^*,\dot{\theta}^*)} \dot{\psi}_h(\theta,\dot{\theta}) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad (18)$$

Since the differentiability problems only arise on the guard, an in particular only on the Zeno equilibria, the limit in equation (18) suffices for the evaluation in **R**3.

R4: Let $(\theta,\dot{\theta}) \in G_h$. Then $h(\theta) = 0$ and $\dot{h}(\theta,\dot{\theta}) \leq 0$. So substitution into equation (16) gives $\psi_h(\theta,\dot{\theta}) = (0, 2|\dot{h}(\theta,\dot{\theta})|)^T$.

Multiplying both sides of equation (15) on the left by $dh(\theta)$ and the definition of $\dot{h}(\theta,\dot{\theta})$ gives $\dot{h}(R_h(\theta,\dot{\theta})) = -e\dot{h}(\theta,\dot{\theta})$.

Therefore $\psi_h(R_h(\theta,\dot{\theta})) = (2e|\dot{h}(\theta,\dot{\theta})|, 0)^T$. So if $\gamma_h^l = \gamma_h^u = e$, **R**4 holds with $\psi_h(R_h(\theta,\dot{\theta}))_1 \in [e\psi_h(\theta,\dot{\theta})_2, e\psi_h(\theta,\dot{\theta})_2]$.

R5: $\left|\gamma_h^u \frac{d\psi_h(\theta^*,\dot{\theta}^*)_2 f_L(\theta^*,\dot{\theta}^*)}{d\psi_h(\theta^*,\dot{\theta}^*)_1 f_L(\theta^*,\dot{\theta}^*)}\right| = \left|e\frac{1}{-1}\right| = e < 1.$

R6: Take a point $(\theta,\dot{\theta}) \in G_h \cap V$ (this is the only step that requires a norm, and hence the coordinate chart on V). We upper bound the growth due to the reset map as follows,

$$\begin{aligned} &\|R_h(\theta,\dot{\theta}) - (\theta^*,\dot{\theta}^*)\| \\ &= \Bigg\|(\theta,\dot{\theta}) - (\theta^*,\dot{\theta}^*) - \\ &\quad \left(0, (1+e)\frac{dh(\theta)\dot{\theta}}{dh(\theta)M(\theta)^{-1}dh(\theta)^T} M(\theta)^{-1} dh(\theta)^T\right)\Bigg\| \\ &\leq \|(\theta,\dot{\theta}) - (\theta^*,\dot{\theta}^*)\| + \\ &\quad (1+e)\frac{|dh(\theta)\dot{\theta}|}{dh(\theta)M(\theta)^{-1}dh(\theta)^T}\|M(\theta)^{-1}dh(\theta)^T\|. \end{aligned}$$

Recall that $\psi_h(\theta,\dot{\theta})_2 = 2|dh(\theta)\dot{\theta}|$. Plugging in the definition of K proves **R**6:

$$\|R_h(\theta,\dot{\theta}) - (\theta^*,\dot{\theta}^*)\| \leq \|(\theta,\dot{\theta}) - (\theta^*,\dot{\theta}^*)\| + K\psi_h(\theta,\dot{\theta})_2.$$

Since **R**1-**R**6 hold, Theorem 3 implies that there is a neighborhood W of $(\theta^*,\dot{\theta}^*)$ with $W \subset V$ such that there is a unique Zeno execution with $c_0(0) = x$ for all $x \in W$. ■

Example 5 (Ball on a Circle): With Zeno stability tools in hand, we revisit the ball bouncing on a circle from Examples 1, 3 and 4. The conditions in equation (3) for Zeno stability follow from the application of Theorem 4.

C. Application to Knee-bounce in Bipedal Robotic Walking

Mechanical knees are an important component of achieving natural and "human-like" walking in bipedal robots [39]. Mechanical "knee-caps", i.e., mechanical stops, are typically added to these mechanical knees to prevent the leg from hyper-extending. Yet with this benefit comes a cost: *knee-bounce*, which occurs when the shin bounces off the mechanical stop repeatedly as the leg attempts to lock. It was experimentally shown by McGeer that this "bouncing" behavior can destabilize the robot in certain situations [40]. This naturally raises the question: will knee-bounce always destabilize robotic walking, or are small amounts of knee-bounce acceptable? To address this question, knee-bounce can be formulated as Zeno behavior and analyzed using the results of this paper. In particular, this subsection summarizes the results of [41], which applied Theorems 3 and 4 of this paper to characterize the effect that knee-bounce has on the existence of walking gaits for bipedal robots with mechanical knees.[3]

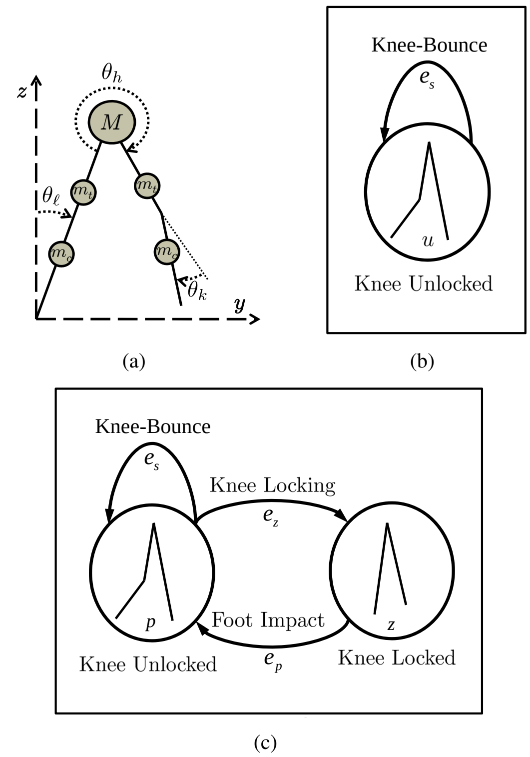


Fig. 3. The model of a biped with knees (a) the Lagrangian hybrid system model $\mathscr{H}_{\mathbf{R}}$ (b) and the generalized completion of this model $\overline{\mathscr{H}}_{\mathbf{R}}$ (c).

Consider a bipedal robot with knees as shown in Figure 3(a). We begin by considering the hybrid Lagrangian associated with this robot, given by $\mathbf{R} = (\Theta_{\mathbf{R}}, L_{\mathbf{R}}, h_{\mathbf{R}})$, where: $\Theta_{\mathbf{R}}$ is the configuration space of the robot with coordinates $\theta = (\theta_l, \theta_h, \theta_k)$, where θ_l is the angle of the leg from vertical, θ_h is the angle of the hip and θ_k is the angle of the knee, $L_{\mathbf{R}}$ is the Lagrangian of the robot computed in the standard way [42], and $h_{\mathbf{R}}$ is the constraint that ensures that the knee does not hyperextend, i.e., $h_{\mathbf{R}} = \theta_k$. From this hybrid Lagrangian we obtain a Lagrangian hybrid system:

$$\mathscr{H}_{\mathbf{R}} = (\Gamma_{\mathbf{R}} = (u, e_s = (u,u)), D_{\mathbf{R}}, G_{\mathbf{R}}, R_{\mathbf{R}}, F_{\mathbf{R}}),$$

which models the robot with a single impact that occurs when the knee "strikes" as shown in Figure 3(b). The Zeno equilibria for this system are given by:

$$Z_{\mathbf{R}} = \{(\theta,\dot{\theta}) \in T\Theta_{\mathbf{R}} : f_{\mathbf{R}}(\theta,\dot{\theta}) \neq 0,\ \ \theta_k = 0,\ \ \dot{\theta}_k = 0\}.$$

[3]A more formal treatment is given in [41].

That is, the set of points where the knee angle is zero with zero velocity, i.e., the set of points where the leg is straight and the knee is "locked." Moreover, it is easy to verify that $\ddot{h}_{\mathbf{R}} < 0$ for a large subset of this set, and thus there are stable Zeno equilibria. Physically, the existence of these stable Zeno equilibria imply that knee-bounce will occur (formally verifying the experimental behavior witnessed by McGeer).

As a result of these stable Zeno equilibria, it is necessary to *complete* the Lagrangian hybrid system model $\mathscr{H}_{\mathbf{R}}$ to allow for solutions to continue after knee locking. The details of this completion process for this system can be found in [41], but to summarize one obtains a new hybrid system $\overline{\mathscr{H}}_{\mathbf{R}}$ graphically illustrated in Figure 3(c) where a "post-Zeno" domain is added where the leg is locked, transitions to that domain occur when the set $Z_{\mathbf{R}}$ is reached, and transitions back to the original "pre-Zeno" domain occurs at foot-strike with reset map being the standard impact equations considered in the bipedal robotics literature [37]. In the case of perfectly plastic impacts at the knee (when $e = 0$ for $R_{\mathbf{R}}$ as computed with (15)), this completed model is the standard model of a bipedal robot with knees that lock [43]. What is of interest is when the assumption of perfectly plastic impacts at the knee is relaxed, and knee-bounce occurs.

It was proven in [41] using Theorem 3 and Theorem 4 that if there exists a locally exponentially stable *plastic periodic orbit* for $\overline{\mathscr{H}}_{\mathbf{R}}$, i.e., a periodic orbit with $e = 0$, and if $\ddot{h}_{\mathbf{R}}(\theta^*, \dot{\theta}^*) < 0$ for $(\theta^*, \dot{\theta}^*)$ a Zeno equilibrium point that is a fixed point of this plastic periodic orbit, then for $e > 0$ sufficiently small there exists a *Zeno periodic orbit* for $\overline{\mathscr{H}}_{\mathbf{R}}$, i.e., a periodic orbit which contains a Zeno execution. In simple terms, this result simply states that if there exists bipedal robotic walking for the assumption of perfectly plastic impacts at the knees, there there will exist robotic walking for small amounts of knee-bounce. To apply this result, we begin by producing a walking gait with plastic impacts at the knees, the trajectories for which can be seen in Figure 4(a). The exponential stability of the periodic orbit associated with this walking gait can be checked by numerically computing the Poincaré map at the fixed point $(\theta^*, \dot{\theta}^*)$, and we can check that the Zeno equilibrium point $(\theta^*, \dot{\theta}^*)$ is Zeno stable by noting that $\ddot{h}_{\mathbf{R}}(\theta^*, \dot{\theta}^*) = -50.135 < 0$. Thus there will be walking even with knee bounce as long as it is sufficiently small. In fact, we find that even taking $e = 0.25$ there is still a stable walking gait; the trajectories of this walking gait can be seen in Figure 4(b).

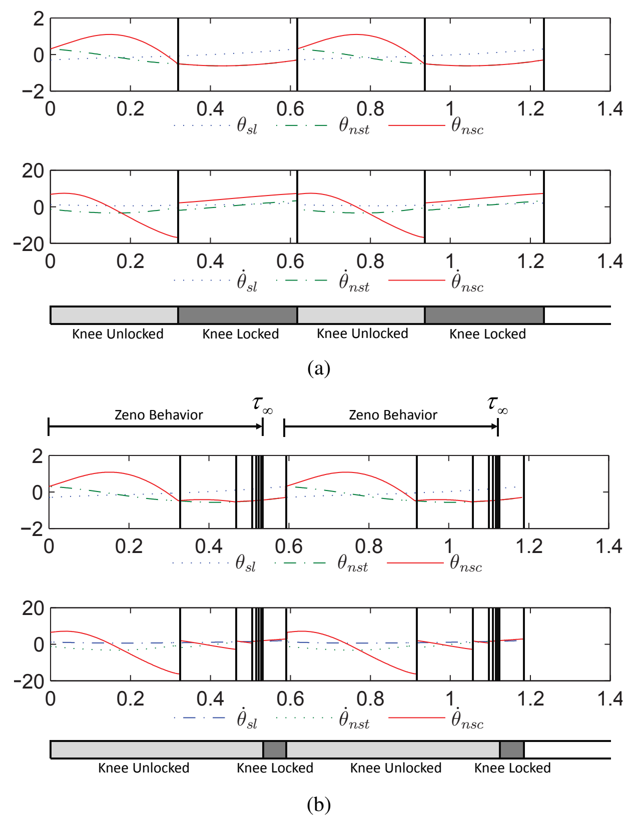


Fig. 4. The trajectories of the bipedal robot for a walking gait in the case of plastic impacts at the knee (a) and knee-bounce, or Zeno behavior, at the knee (b).

V. Conclusion

In this paper, we developed Lyapunov-like sufficient conditions for Zeno stability. The proof methodology had two main components. First, we defined a class of hybrid systems with simple conditions for Zeno stability. Then, we proposed special structured (Lyapunov-like) functions that map executions of interesting hybrid systems to executions of the simple Zeno hybrid systems on the first quadrant of the plane.

Our Lyapunov-like theorem applies equally well to isolated and non-isolated Zeno equilibria. Covering both cases was necessary, since we observed that a stable Zeno equilibrium displayed asymptotic stability if and only if it was isolated. Furthermore, since most interesting Lagrangian hybrid systems only have non-isolated Zeno equilibria, the study of the stability of non-isolated Zeno equilibria is fundamental. Because most of the existing conditions for Zeno behavior required either isolated Zeno equilibria or asymptotically stable Zeno equilibria, they all had similar limitations.

Our applications to Lagrangian hybrid systems showed that our sufficient conditions for Zeno stability can handle some non-trivial, high dimensional hybrid systems. Furthermore, the Lyapunov-like sufficient conditions specialize to algebraic constraints on the Zeno equilibria. In particular, in Lagrangian hybrid systems, we can infer Zeno stability properties based on the zeroth-order approximation to the vector fields at the Zeno equilibria, similar to the local approximation results of [1], [3], [5].

Future work on Zeno stability must push the theory towards real-world systems. As summarized in Subsection IV-C, the theory from this paper has been used to characterize knee-bounce phenomena in bipedal robotic systems. More work is needed to extend the theory for reasoning about practical systems from robotics, control, and verification in which Zeno behavior occurs in the models.

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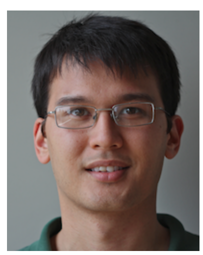

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