Continuity and Smoothness Properties of Nonlinear Optimization-Based Feedback Controllers

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Abstract— **Online optimization-based controllers are becoming increasingly prevalent as a means to control complex high-dimensional nonlinear systems, e.g., bipedal and humanoid robots, due to their ability to balance multiple control objectives subject to input constraints. Motivated by these applications, the goal of this paper is to explore the continuity and smoothness properties of feedback controllers that are formulated as quadratic programs (QPs). We begin by drawing connections between these optimization-based controllers and a family of perturbed nonlinear programming problems commonly studied in operations research. With a view towards robotic systems, some existing results on perturbed nonlinear programming problems are extended and specialized to address conditions that arise when quadratic programs are used to enforce the convergence of control Lyapunov functions (CLFs).**

The main result of this paper is a novel set of conditions on the continuity of QPs that can be used when a subset of the constraints vanishes. A simulation study of position regulation in the compass gait biped demonstrates how the new conditions of this paper can be applied to more complex robotic systems.

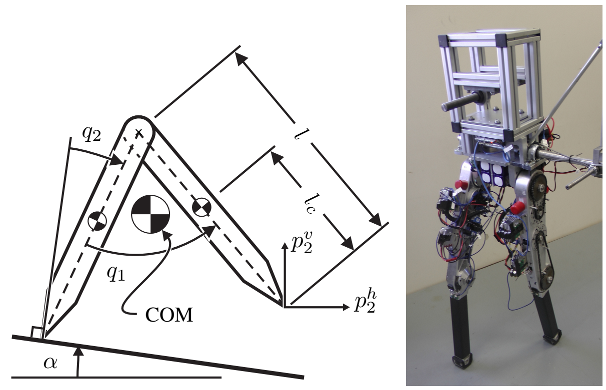

Fig. 1. (Left) The planar two-link compass-gait walker is a simple robot that can be used to illustrate the smoothness properties of feedback controllers implemented via constrained optimization problems (nonlinear programs). (Right) The bipedal robot DURUS is a complex nonlinear system; working to implement the control-via-optimization methods on DURUS and similar robots has motivated the development and understanding of the formal results presented.

I. INTRODUCTION

With the rise in computation power and the ubiquity of powerful and inexpensive mobile processors, there is an ever increasing ability to realize implicitly defined optimization-based controllers in real-time. The benefits of controllers of this form have been known for some time [19], especially in the context of model predictive control (MPC) of linear systems [13], [14] (and, in limited scope, nonlinear systems [16]). Yet the increase in mobile computation power points to the ability to extend these benefits to the real-time control of general nonlinear systems where the feedback controller itself is implemented as the solution to a nonlinear optimization problem, i.e., a nonlinear programming problem, at each time step. The ability to characterize controllers of this form would have a variety of unique benefits: a large number of control objectives could be simultaneously considered, physical constraints (e.g., torque bounds and state constraints) could be dynamically balanced against these control objectives, and trajectory planning could be subsumed into the real-time control process. These advantages are especially acute in the context of nonlinear systems, for which robotic systems provides a quintessential example.

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There has been a recent surge in the use of optimization-based feedback controllers, and specifically quadratic program based (QP) controllers, in the robotics domain due to the aforementioned benefits. Beginning at the level of center of mass and foot placement planning, the fact that under certain stringent conditions the center of mass dynamics become linear allows for the formulation of a QP based planner [20]. This observation can be coupled with the fact that the equations of motion are affine in torque to achieve whole-body behaviors on humanoid robots [12], [21], [22], [24]. Coming from the perspective of nonlinear control of robotic systems, QP controllers can also be formulated to exploit the fact that control Lyapunov functions (CLF) for these systems are, again, affine in torque [9]. This framework was utilized, with the addition of torque bounds, to realize bipedal robotic walking [3], [5], [10] (see videos of the behavior at [1]). Building upon this observation, this methodology was extended to allow for the balancing of multiple control objectives subject to force and torque constraints [2], was recently realized onboard and in real-time on prosthesis [25], and has been utilized to achieve robotic walking via nonlinear MPC related concepts [17]. Yet in all of the above applications, the authors have found few formal results on the continuity and smoothness properties of these optimization-based controllers. Except in special cases, establishing these properties is a difficult problem due to the lack of a closed from expression for the resulting controllers.

In this paper we investigate the smoothness properties of feedback controllers that are based on convex quadratic programs. The first contribution of this paper is to create a connection point between QP optimization-based controllers and classical results from operations research on the perturbation analysis of nonlinear programs [4]. Motivated by the applications to nonlinear and robotic systems, we then focus on extending existing results to address constraint properties that arise when QPs are used to enforce the convergence of CLFs. This allows for the presentation of the main result of this paper: sufficient conditions for the continuity of QP based controllers at points where control authority vanishes for a subset of the constraints (that is, without the assumption of linear independence of the binding constraints). To illustrate these results, simple examples involving a compass-gait robot [23] are considered, and discontinuities in the controller are discovered by considering the conditions of the main result of the paper.

It is important to note that previous work by the authors [15] considered the Lipschitz continuity of feedback controllers derived from quadratic programs. Yet these conditions were incorrectly stated; the error resulted from an incorrect assumption that a single-valued mapping that is both upper and lower Lipschitz continuous is Lipschitz continuous (in the traditional sense)—an assumption that does not hold, in general. This was discovered by the authors through the Robinson's well-known example illustrating loss of Lipschitz continuity [18] which is a counterexample to the main theorem presented in the previous paper [15]. This paper corrects [15] while simultaneously extending the results to consider constraint conditions that are more representative of CLF control of bipedal locomotion.

II. Background

A general, nonlinear programming problem $\mathcal{P}$ can be written in the following form,

$$\mathcal{P} = \begin{cases} \text{argmin}_{x \in R^n} & C(x) \\ \text{subject to} & g_i(x) \leq 0 \quad \forall\ i \in \{1 \ldots p\} \\ & h_i(x) = 0 \quad \forall\ i \in \{1 \ldots q\} \end{cases} \tag{1}$$

where for some $n > 0$, $p \geq 0$, $q \geq 0$ we have $C : R^n \to R$, $g_i : R^n \to R$ for all $i \in \{1 \ldots p\}$, $h_i : R^n \to R$ for all $i \in \{1 \ldots q\}$. For this paper, a *solution* of $\mathcal{P}$ is any point (x, λ, w) where $x \in R^n$ is a minimizer of $\mathcal{P}$ with associated vectors of Lagrange multipliers $\lambda \in R^p$ and $w \in R^q$ corresponding to enforcement of the inequality and equality constraints, respectively. In general we will not assume that a solution exists or is unique unless we explicitly make such an assumption.

The nonlinear program above can be subject to a general-form vector perturbation $\epsilon \in R^m$, $m > 0$ as follows

$$\mathcal{P}(\epsilon) = \begin{cases} \text{argmin}_{x \in R^n} & C(x, \epsilon) \\ \text{subject to} & g_i(x, \epsilon) \leq 0 \quad \forall\ i \in \{1 \ldots p\} \\ & h_i(x, \epsilon) = 0 \quad \forall\ i \in \{1 \ldots q\} \end{cases} \tag{2}$$

where the respective domains of C, g_i, and h_i are modified as appropriate. The problem above can be viewed as a point-to-set mapping between vector parameters $\epsilon \in R^m$ and solution sets $\mathcal{X} \subset R^{n+p+q}$.

Given a control system of the form $\dot{x} = f(x, u)$, $x \in R^m$, a feedback control law can be formulated as a perturbed nonlinear programming problem as follows

$$\mathcal{P}(x) = \begin{cases} \text{argmin}_{u \in R^n} & C(u, x) \\ \text{subject to} & g_i(u, x) \leq 0 \quad \forall\ i \in \{1 \ldots p\} \\ & h_i(u, x) = 0 \quad \forall\ i \in \{1 \ldots q\} \end{cases} \tag{3}$$

where control values u are selected from the set of minimizers. Although it is only a simple change of variable names between (2) and (3), the comparison illustrates how the analysis of optimization-based controllers (3) can build on existing literature on the sensitivity analysis of perturbed nonlinear programming problems (2).

An alternative approach to analyzing the smoothness of minimizers of (3) would be to transform the convex program into a form where the pointwise min-norm controller [9] could be applied. Under conditions provided in [9], Section 4.2, the min-norm controller is known to exist and to be continuous. However, if the perturbed nonlinear program (3) cannot be put into the form of the min-norm controller, or if a result stronger than continuity is needed, existing analysis such as Fiacco's theorem on continuous differentiability (presented in Section III-A) could potentially be used.

A. Literature on Perturbation Analysis of Nonlin. Programs

An early and influential book on stability analysis of nonlinear programming is due to Fiacco and McCormick [8], providing detailed analysis of directional perturbations of an isolated minimizer. In [7] Fiacco generalizes the method to arbitrary perturbations, provides a set of sufficient conditions for the derivative to exist, and gives a closed form expression for the derivative when the conditions are met. Robinson constructs an interesting counterexample in [18], showing that without linear independence of the binding constraints, the solution to a particular nonlinear programming program is not necessarily Lipschitz continuous. Jittorntrum provides a later analysis in [11] proving directional differentiability without strict complementary slackness. An extensive survey of stability and continuity results of nonlinear programming under abstract conditions was assembled by Bonnans and Shapiro in [4]. The survey work of Fiacco and Ishizuka [6] is an excellent introduction to perturbation analysis and the scope of the results that are available under traditional assumptions.

B. Additional Definitions

The standard Karush-Kuhn-Tucker (KKT) conditions can be derived by applying the method of Lagrange multipliers to the problem $\mathcal{P}(x)$ as in (3) where the Lagrangian is

$$\mathcal{L}(x, u, \lambda, w) = C(u, x) + \sum_{i=1}^{p} \lambda_i g_i(u, x) + \sum_{i=1}^{q} w_i h_i(u, x). \tag{4}$$

A point (u^*, λ^*, w^*) meets the *KKT conditions* for $\mathcal{P}(x^*)$ if

$$\nabla \mathcal{L}(x^*, u^*, \lambda^*, w^*) = 0 \tag{5}$$

and

$$\begin{array}{rcll} h_i(u^*,x^*) & = & 0 & \text{for } i\in\{1\ldots q\} \\ g_i(u^*,x^*) & \le & 0 & \text{for } i\in\{1\ldots p\} \\ \lambda_i^* g_i(u^*,x^*) & = & 0 & \text{for } i\in\{1\ldots p\} \\ \lambda_i^* & \ge & 0 & \text{for } i\in\{1\ldots p\}. \end{array} \tag{6}$$

A point (u^*,λ^*,w^*) satisfies the *linear independence constraint qualification (LICQ)* for $\mathcal{P}(x^*)$ (or, in different terminology, is said to be *regular*) if the gradients of the active constraints of $\mathcal{P}(x^*)$ are linearly independent. That is, the matrix

$$\left[\begin{array}{l} \frac{\partial}{\partial u} h_i(u^*,x^*) \;\; \forall \;\; i\in\{1\ldots q\} \\ \frac{\partial}{\partial u} g_i(u^*,x^*) \;\; \forall \;\; i \text{ s.t. } g_i(u^*,x^*)=0 \end{array}\right] \tag{7}$$

has full row rank.

A point (u^*,λ^*,w^*) satisfies *complementary slackness* for $\mathcal{P}(x^*)$ if $\lambda_i^* g_i(u^*,x^*)=0$ for each i and satisfies *strict complementary slackness* if there does not exist any i for which both $\lambda_i^*=0$ and $g_i(u^*,x^*)=0$.

A point (u^*,λ^*,w^*) satisfies the *second order sufficient conditions* for $\mathcal{P}(x^*)$ if

i) the functions $C(u,x)$, $g_i(u,x)$ for $i\in\{1\ldots p\}$, and $h_i(u,x)$ for $i\in\{1\ldots q\}$ are twice continuously differentiable in u, for u in an open neighborhood of u^* and $x=x^*$ fixed
ii) the point (u^*,λ^*,w^*) satisfies the KKT conditions for $\mathcal{P}(x^*)$
iii) for any $y\neq 0$ satisfying
   $y^T\frac{\partial}{\partial u}g_i(u^*,x^*)\le 0$ for all i where $\lambda_i^*=0$
   $y^T\frac{\partial}{\partial u}g_i(u^*,x^*)=0$ for all i where $\lambda_i^*>0$
   $y^T\frac{\partial}{\partial u}h_i(u^*,x^*)=0$ for $i\in\{1\ldots p\}$
   the following inequality holds:

$$y^T\nabla^2\mathcal{L}(x^*,u^*,\lambda^*,w^*)y>0,$$

   where the Hessian matrix operator, denoted ∇^2, is taken with respect to the variable u.

A function $f: R^m\to R^n$ is *Lipschitz continuous* at $x\in R^m$ if there exists values $\delta>0$ and $L>0$, both perhaps dependent on the value of x, such that for all $x_1,x_2\in B_\delta(x)\subset R^m$, $\|f(x_2)-f(x_1)\|\le L\|x_2-x_1\|$.

III. Two Results from the Literature on Perturbation Analysis of Convex Programs

Perturbed convex nonlinear programs such as (2) have been analyzed extensively in the literature. This section reviews two existing results that are relevant in analyzing (3) as a feedback controller. The first result below provides conditions under which the minimizer is unique and differentiable. The second provides an example showing that under a different set of conditions, a unique minimizer is not necessary Lipshitz continuous.

A. Sufficient Conditions for Differentiability

The following theorem is from Fiacco [7] Theorem 2.1 and has been modified to fit the notation of this paper.

Theorem 1: [Differentiability of the Minimizer] Given a perturbed nonlinear programming problem $\mathcal{P}(x)$ as in (3) and a parameter vector $x^*=0$, suppose the following

H1.1) the functions $C(u,x)$, $g_i(u,x)$ for $i\in\{1\ldots p\}$, and $h_i(u,x)$ for $i\in\{1\ldots q\}$ are twice continuously differentiable in (u,x) in an open neighborhood of $(u^*,0)$
H1.2) there exists a point (u^*,λ^*,w^*) that satisfies the second order sufficient condition of $\mathcal{P}(x)$ at $x^*=0$
H1.3) the point (u^*,λ^*,w^*) satisfies the linear independence constraint qualification of $\mathcal{P}(x)$ at $x^*=0$
H1.4) the point (u^*,λ^*,w^*) satisfies strict complementary slackness for $\mathcal{P}(x)$ at $x^*=0$

then

a) u^* is an isolated local minimizer of $\mathcal{P}(x)$ at $x^*=0$
b) for x in a neighborhood of 0 there exist continuously differentiable functions $u(x)$, $\lambda(x)$, and $w(x)$ (with $u(0)=u^*$, $\lambda(0)=\lambda^*$, and $w(0)=w^*$) such that $u(x)$ is an isolated local minimizer of $\mathcal{P}(x)$, with associated unique Lagrange multiplier vectors $\lambda(x)$ and $w(x)$
c) the point $(u(x),\lambda(x),w(x))$ satisfies strict complementary slackness and the linear independence constraint qualification for $\mathcal{P}(x)$ for x in a neighborhood of 0

B. Robinson's Counterexample

The following example, presented by Robinson in [18], shows that even under strict convexity, a perturbed quadratic program might result in solutions that are non-Lipschitz with respect to perturbations.

Consider the following strictly convex quadratic program

$$\mathcal{P}(x)=\left\{\begin{array}{ll} \operatorname{argmin}_{u\in R^4} & \frac{1}{2}u^Tu \\ \text{subject to} & A(x)u\ge b(x) \end{array}\right. \tag{8}$$

where $x=(x_1,x_2)\in R^2$ and

$$A(x)=\left[\begin{array}{cccc} 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & x_1 \end{array}\right] \quad b(x)=\left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1+x_2 \end{array}\right]. \tag{9}$$

Note that for $0<x_1,\;\; 0\le x_2\le\frac{1}{2}x_1^2$, the unique minimizer can be computed in closed form

$$u(x)=(0,0,1,0)^T+\frac{x_2}{x_1}\;(0,0,0,1)^T, \tag{10}$$

and for $x_1=x_2=0$ the solution is

$$u(x)=(0,0,1,0)^T. \tag{11}$$

Although we have existence and uniqueness of the minimizer $u(x)$ for any value of $x=(x_1,x_2)$, the minimizer itself is not Lipschitz continuous in any open set containing $(x_1,x_2)=(0,0)$.

IV. Application to Feedback via Control Lyapunov Functions

Given an affine nonlinear control system

$$\dot{x} = f(x) + g(x)u \tag{12}$$

with $x \in R^m$, $u \in R^n$, consider a set of control objectives, each encoded as the zeroing of a smooth scalar output function $y_i : R^m \to R$.

A. Input-Output Linearization and RES-CLFs

If we have a set of n outputs and the corresponding vector $y(x) = (y_1(x), \ldots, y_n(x))^T$ has a well-defined vector relative degree, then feedback linearization can be applied. Assume that our vector of outputs has vector relative degree 2. The control law

$$u(x) = L_g L_f y(x)^{-1}(\mu - L_f^2 y(x)) \tag{13}$$

when applied to the dynamics (12) results in the linear decoupled input-output system

$$\ddot{y} = \mu \tag{14}$$

for any $\mu \in R^n$. Or, equivalently

$$\dot{\eta} = \underbrace{\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}}_{F} \eta + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{G} \mu. \tag{15}$$

for $\eta = (y^T, \dot{y}^T)^T$. With this form, we can now construct a RES-CLF following the steps given in [3], [5].

B. Constructing a Rapidly Exponentially Stabilizing Control Lyapunov Function (RES-CLF)

For the system in (15), a continuously differentiable function $V_\varepsilon : X \to \mathbb{R}$ is a *rapidly exponentially stabilizing control Lyapunov function (RES-CLF)* if there exist positive constants $c_1, c_2, c_3 > 0$ such that for all $0 < \varepsilon < 1$,

$$c_1 \|\eta\|^2 \le V_\varepsilon(\eta) \le \frac{c_2}{\varepsilon^2}\|\eta\|^2, \tag{16}$$

$$\inf_{\mu \in \mathbb{R}^n} \left[L_F V_\varepsilon(\eta) + L_G V_\varepsilon(\eta)\mu + \frac{c_3}{\varepsilon} V_\varepsilon(\eta) \right] \le 0, \tag{17}$$

for all $\eta \in X$. A RES-CLF for the outputs η can be constructed via:

$$V_\varepsilon(\eta) := \eta^T \underbrace{I_\varepsilon P I_\varepsilon}_{P_\varepsilon} \eta, \quad I_\varepsilon := \mathrm{diag}\left(\frac{1}{\varepsilon} I, I\right), \tag{18}$$

where I is the identity matrix and $P = P^T > 0$ solves the the continuous time algebraic Riccati equations (CARE) $F^T P + PF - PGG^T P + Q = 0$ for $Q = Q^T > 0$. The time-derivative of (18) is given by

$$\dot{V}_\varepsilon(\eta) = L_F V_\varepsilon(\eta) + L_G V_\varepsilon(\eta)\mu, \tag{19}$$

where

$$L_F V_\varepsilon(\eta) = \eta^T (F^T P_\varepsilon + P_\varepsilon F)\eta, \tag{20}$$

$$L_G V_\varepsilon(\eta) = 2\eta^T P_\varepsilon G, \tag{21}$$

are the Lie derivatives of $V_\varepsilon(\eta)$ along the vector fields F and G. As noted in [3] a RES-CLF can be realized by including the inequality constraint (17) in a feedback controller based on nonlinear programming.

$$\mathcal{P}(\eta) = \begin{cases} \mathrm{argmin}_{\mu \in R^n} \ \frac{1}{2}\mu^T \mu \\ \text{subject to} \quad L_G V_\varepsilon(\eta)\mu \le -\frac{c_3}{\varepsilon} V_\varepsilon(\eta) - L_F V_\varepsilon(\eta) \end{cases} \tag{22}$$

C. Conditions for Continuously Differentiable Minimizers of the CLF-QP

The following theorem provides sufficient conditions for the unique minimizer of (22) to be continuously differentiable

Theorem 2: Consider the parameterized nonlinear program (22) at $\eta = \eta^*$. Suppose that for this vector the following hold:

H2.1) $L_G V_\varepsilon(\eta)$, $V_\varepsilon(\eta)$, $L_F V_\varepsilon(\eta)$ and are twice continuously differentiable in an open neighborhood containing η^*
H2.2) there exists a point (μ^*, λ^*, w^*) that satisfies the linear independent constraint qualification of $\mathcal{P}(\eta^*)$.
H2.3) the point (μ^*, λ^*, w^*) satisfies complementary slackness holds for $\mathcal{P}(\eta^*)$

Then there exists a continuously differentiable function $\mu(\eta)$ defined on an open neighborhood containing η^* such that for all η in this neighborhood $\mu(\eta)$ is the unique minimizer of $\mathcal{P}(\eta)$.

Proof: Note that the trivial quadratic cost function implies that the second order sufficient condition holds, and apply Theorem 1. ∎

Note that for a CLF-QP as in (22) the theorem above shows that the solution $\mu(\eta)$ is continuously differentiable whenever $L_G V_\varepsilon(\eta)$ has full row rank and strict complementary slackness is satisfied. Most notably, this breaks down at $V(\eta) = \eta^T P_\varepsilon \eta = 0$, when $\eta = 0$ and $L_G V_\varepsilon(\eta) = 0$. The convergence constraint becomes

$$L_G V_\varepsilon(0)\mu \le -\frac{c_3}{\varepsilon} V_\varepsilon(0) - L_F V_\varepsilon(0) \tag{23}$$

or, equivalently

$$0\mu = 0. \tag{24}$$

The vector of zeros means that the linear independent constraint qualification breaks down. The main theorem of the paper which is presented in the following section provides an alternative set of conditions that does not require the linear independent constraint qualification, and can thus be used to analyze constraints such as (23).

V. Continuity Under Vanishing Constraints

This section contains the main result of the paper, establishing sufficient conditions under which continuity of the minimizer (but not necessarily Lipschitz continuity or differentiability) can be recovered when the linear independent constraint qualification does not hold. To demonstrate the application of the main theorem, Section VI provides a simple case study under which the conditions Theorem 1 fail, but the continuity of the minimizer can still be proven using the main theorem (Theorem 3).

A. Properties of Solutions to Nonlinear Programs

The following propositions provide useful relationships between nonlinear programs that share the same objective function.

Proposition 1: Given a pair of nonlinear programs

$$\mathcal{P}_0 = \begin{cases} \text{argmin} & f(u) \\ \text{subject to} & u \in S_0 \end{cases} \tag{25}$$

$$\mathcal{P}_1 = \begin{cases} \text{argmin} & f(u) \\ \text{subject to} & u \in S_1 \end{cases} \tag{26}$$

with $S_0 \subset S_1 \subset D \subset R^n$, $n \geq 1$, $f : R^n \to R$, assume that $\mathcal{P}_0$ and $\mathcal{P}_1$ have minimizers (that are possibly non-unique), designated u_0^* and u_1^* respectively. Then $f(u_0^*) \geq f(u_1^*)$

Proof: Suppose the alternative, that under the given assumptions $f(u_0^*) < f(u_1^*)$. Then, u_1^* could not be a minimizer of $\mathcal{P}_1$ because $u_0^* \in S_0 \subset S_1$ is a feasible point of $\mathcal{P}_1$ with a lower value for the objective function than u_1^* provides. However, it is given that u_1^* is a minimizer. Thus the alternative must be false, thus proving the claim by contradiction. ∎

Proposition 2: Given three nonlinear programs:

$$\mathcal{P}_\text{S} = \begin{cases} \text{argmin} & f(u) \\ \text{subject to} & u \in \Phi_\text{S} \end{cases} \tag{27}$$

$$\mathcal{P} = \begin{cases} \text{argmin} & f(u) \\ \text{subject to} & u \in \Phi \end{cases} \tag{28}$$

$$\mathcal{P}_\text{B} = \begin{cases} \text{argmin} & f(u) \\ \text{subject to} & u \in \Phi_\text{B} \end{cases} \tag{29}$$

with $\Phi_\text{S} \subset \Phi \subset \Phi_\text{B} \subset D \subset R^n, n \geq 1$ where $f : D \to R$, suppose that

i) $\mathcal{P}_\text{S}$, $\mathcal{P}$ and $\mathcal{P}_\text{B}$ have unique minimizers, designated u_S^*, u^*, and u_B^* respectively.

ii) There exists some $c > 0$ such that for any $v_1, v_2 \in D$,

$$c\|v_1 - v_0\|^2 \leq |f(v_1) - f(v_0)| \tag{30}$$

iii) The function f is Lipschitz continuous on D. That is, there exists $L \geq 0$ such that for any $v_0, v_1 \in D$,

$$|f(v_1) - f(v_0)| \leq L\|v_1 - v_0\| \tag{31}$$

Then

$$\|u^* - u_\text{S}^*\|^2 \leq \frac{L}{c}\|u_\text{S}^* - u_\text{B}^*\|. \tag{32}$$

Proof: Apply Proposition 1 twice to see

$$f(u_\text{S}^*) \geq f(u^*) \geq f(u_\text{B}^*). \tag{33}$$

By the second order growth bound (ii) we have that

$$c\|u_\text{S}^* - u^*\|^2 \leq f(u_\text{S}^*) - f(u^*) \tag{34}$$

Note that by (33)

$$f(u_\text{S}^*) - f(u^*) = f(u_\text{S}^*) - f(u_\text{B}^*) + \underbrace{f(u_\text{B}^*) - f(u^*)}_{<0} \tag{35}$$

So that (34) becomes

$$c\|u_\text{S}^* - u^*\|^2 \leq f(u_\text{S}^*) - f(u_\text{B}^*). \tag{36}$$

Applying the Lipschitz property (iii) to the above leads to

$$\|u^* - u_\text{S}^*\|^2 \leq \frac{L}{c}\|u_\text{S}^* - u_\text{B}^*\|. \tag{37}$$

which completes the proof. ∎

B. Additional definitions

Suppose that a given parameterized nonlinear program

$$\mathcal{P}(x) = \begin{cases} \text{argmin} & \frac{1}{2}u^T H(x) u + c(x)^T u \\ \text{subject to} & u \in \Phi(x) \end{cases} \tag{38}$$

with

$$\Phi(x) = \left\{ u \in R^n \,\middle|\, \begin{array}{r} A_{\text{EQ},i}(x)u = b_{\text{EQ},i}(x),\ i \in \{1\dots q\} \\ A_i(x)u \leq b_i(x), \quad i \in \{1 \dots p\} \end{array} \right\}$$

has a unique minimizer u at a point of interest x. Using this minimizer, define parameter dependent index sets for the vanishing constraints, the non-vanishing active constraints, and the non-vanishing inactive constraints as follows:

$$\begin{aligned} I_\text{V}(x) &= \{i \in \{1\dots p\} \mid A_i(x) = 0\} \\ I_\text{A}(x) &= \left\{ i \in \{1\dots p\} \,\middle|\, \begin{array}{c} A_i(x) \neq 0, \\ A_i(x)u = b_i(x) \end{array} \right\} \\ I_\text{I}(x) &= \left\{ i \in \{1\dots p\} \,\middle|\, \begin{array}{c} A_i(x) \neq 0, \\ A_i(x)u < b_i(x) \end{array} \right\} \end{aligned} \tag{39}$$

Let the matrices $A_\text{A}(x, x^*)$ and $A_\text{V}(x, x^*)$ be defined as

$$A_\text{A}(x,x^*) = \begin{bmatrix} \vdots \\ A_i(x) \\ \vdots \end{bmatrix}, b_\text{A}(x,x^*) = \begin{bmatrix} \vdots \\ b_i(x) \\ \vdots \end{bmatrix} \quad i \in I_\text{A}(x^*) \tag{40}$$

$$A_\text{V}(x,x^*) = \begin{bmatrix} \vdots \\ A_i(x) \\ \vdots \end{bmatrix}, b_\text{V}(x,x^*) = \begin{bmatrix} \vdots \\ b_i(x) \\ \vdots \end{bmatrix} \quad i \in I_\text{V}(x^*), \tag{41}$$

With these constructions we will define a new program

$$\mathcal{P}_\text{B}(x,x^*) = \begin{cases} \text{argmin}_{u \in R^n} & \frac{1}{2}u^T H(x) u + c(x)^T u \\ \text{subject to} & A_\text{B}(x,x^*)u = b_\text{B}(x,x^*) \end{cases} \tag{42}$$

where

$$A_\text{B}(x,x^*) = \begin{bmatrix} A_\text{EQ}(x) \\ A_\text{A}(x,x^*) \end{bmatrix}, b_\text{B}(x,x^*) = \begin{bmatrix} b_\text{EQ}(x) \\ b_\text{A}(x,x^*) \end{bmatrix}. \tag{43}$$

Define yet another new program

$$\mathcal{P}_\text{S}(x,x^*) = \begin{cases} \text{argmin}_{u \in R^n} & \frac{1}{2}u^T H(x) u + c(x)^T u \\ \text{subject to} & A_\text{S}(x,x^*)u = b_\text{S}(x,x^*) \end{cases} \tag{44}$$

where

$$A_\text{S}(x,x^*) = \begin{bmatrix} A_\text{EQ}(x) \\ A_\text{A}(x,x^*) \\ A_\text{V}(x,x^*) \end{bmatrix}, b_\text{S}(x,x^*) = \begin{bmatrix} b_\text{EQ}(x) \\ b_\text{A}(x,x^*) \\ b_\text{V}(x,x^*) \end{bmatrix}.$$

Finally, let

$$\Delta(x, x^*) = u_{\text{S}}(x, x^*) - u_{\text{B}}(x, x^*) \tag{45}$$

We now state the main theorem of the paper, providing conditions under which we can ensure continuity of a QP without assuming the linear independent constraint qualification is met.

Theorem 3: [Main Theorem] A parameterized nonlinear program $\mathcal{P}(x)$ as in (38) will have a unique minimizer $u(x)$ that is continuous at x^* if the following hold:

H3.1) $\Phi(x)$ is non-empty for all $x \in B_r(x^*)$ for some $r > 0$
H3.2) $\mathcal{P}(x)$ is twice continuously differentiable on $B_r(x^*)$
H3.3) $H(x)$ is strictly positive definite on $B_r(x^*)$
H3.4) $A_{\text{B}}(x, x^*)$ has full row rank for $x \in B_r(x^*)$
H3.5) $I_{\text{I}}(x^*) \subset I_{\text{I}}(x)$ and $I_{\text{A}}(x^*) \subset I_{\text{A}}(x)$, $\forall x \in B_r(x^*)$
H3.6) $\lim_{x\to x^*} \Delta(x, x^*) = 0$

Proof: The proof proceeds in two parts, the first showing that $\lim_{x\to x^*} \|u(x) - u_{\text{S}}(x, x^*)\| = 0$ with the second part utilizing this property to show $\lim_{x\to x^*} \|u(x) - u(x^*)\| = 0$. For the first part of the proof, define

$$s(x, x^*) = \frac{1}{2}\left(\min_{i\in I_I(x^*)} (b_i(x) - A_i(x)u_{\text{B}}(x, x^*))\right) \tag{46}$$

so that

$$A_i(x)u_{\text{B}}(x, x^*) + s(x, x^*) < b_i(x), \quad \forall i \in I_I(x^*). \tag{47}$$

Note that for a fixed x^* and for $x \in B_r(x^*)$, $u_{\text{B}}(x, x^*)$ is continuous by H3.1 - H3.5 and Theorem 1. Therefore $s(x, x^*)$ is continuous in x in this same region. Going further, define $\tau(x, x^*)$ as follows:

$$\tau(x, x^*) = \min_{i\in I_I(x^*)} \left(\frac{s(x, x^*)}{\|A_i(x)\|}\right). \tag{48}$$

We know that $A_i(x) \neq 0, i \in I_{\text{I}}(x^*), x \in B_r(x^*)$ because of H3.5 and the definition of the set $I_{\text{I}}(x^*)$. Thus if $\|\Delta(x, x^*)\| < \tau(x, x^*)$, then for each $i \in I_{\text{I}}(x^*)$,

$$\|A_i(x)\| \; \|\Delta(x, x^*)\| < s(x, x^*). \tag{49}$$

By the Cauchy Schwarz inequality, for each $i \in I_{\text{I}}(x^*)$

$$A_i(x)\Delta(x, x^*) \le |A_i(x)\Delta(x, x^*)| \le \|A_i(x)\| \; \|\Delta(x, x^*)\|. \tag{50}$$

Combining the inequalities from (47), (49) and (50) we have that for each $i \in I_{\text{I}}(x^*)$, if $\|\Delta(x, x^*)\| < \tau(x, x^*)$ then

$$\begin{aligned} A_i(x)u_{\text{S}}(x, x^*) &= A_i(x)(u_{\text{B}}(x, x^*) + \Delta(x, x^*)) \\ &\le A_i(x)u_{\text{B}}(x, x^*) + \|A_i(x)\| \; \|\Delta(x, x^*)\| \\ &\le A_i(x)u_{\text{B}}(x, x^*) + s(x, x^*) \\ &< b_i(x). \end{aligned} \tag{51}$$

Because of this and the fact that (by definition)

$$A_{\text{S}}(x, x^*)u_{\text{S}}(x, x^*) = b_{\text{S}}(x, x^*) \tag{52}$$

we can conclude that if $\|\Delta(x, x^*)\| < \tau(x, x^*)$, then $u_{\text{S}}(x, x^*) \in \Phi(x)$.

Notice that the mapping $\tau(x, x^*)$ is well-defined, strictly positive, and continuous in x for all x sufficiently close to x^*. Thus for some $\gamma > 0$, there exists $\tau_{min} > 0$ (perhaps dependent on x^*) such that

$$\tau_{min} \le \min_{x\in B_\gamma(x^*)} \tau(x, x^*). \tag{53}$$

Thus by H3.6, for any x^*, there must exist some $\delta > 0$ (perhaps dependent on the choice of x^*) such that $\|x - x^*\| \le \delta$ implies $\|\Delta(x, x^*)\| \le \tau_{min}$, and therefore

$$\forall x \in B_\delta(x^*), \quad u_{\text{S}}(x, x^*) \in \Phi(x). \tag{54}$$

We know that for $x \in B_r(x^*)$, $\Phi(x) \subset \Phi_{\text{B}}(x, x^*)$ by H3.5 and the construction of $\mathcal{P}_{\text{B}}(x, x^*)$. Apply Proposition 2 with

$$\{u_{\text{S}}(x, x^*)\} \subset \Phi(x) \subset \Phi_{\text{B}}(x, x^*) \tag{55}$$

to conclude that

$$\begin{aligned} \|u(x) - u_{\text{S}}(x, x^*)\|^2 &\le \tfrac{L}{c}\|u_{\text{S}}(x, x^*) - u_{\text{B}}(x, x^*)\| \\ &\le \tfrac{L}{c}\|\Delta(x, x^*)\|. \end{aligned} \tag{56}$$

Take the limit as x approaches x^* and apply H3.6 to complete the first part of the proof.

For the second part of the proof, apply the triangle inequality twice to $\|u(x) - u(x^*)\|$,

$$\begin{aligned} \|u(x) - u(x^*)\| \;\le\; & \|u(x) - u_{\text{S}}(x, x^*)\| \ldots \\ & + \|u_{\text{S}}(x, x^*) - u_{\text{B}}(x, x^*)\| \ldots \\ & + \|u_{\text{B}}(x, x^*) - u(x^*)\| \end{aligned} \tag{57}$$

Now since $A_i(x^*) = 0, \forall i \in I_{\text{V}}(x^*)$, it immediately follows that $u(x^*) = u_{\text{B}}(x^*, x^*)$, and therefore

$$\lim_{x\to x^*} \|u_{\text{B}}(x, x^*) - u(x^*)\| = \lim_{x\to x^*} \|u_{\text{B}}(x, x^*) - u_{\text{B}}(x^*, x^*)\|.$$

By the continuity of $u_{\text{B}}(x, x^*)$, the above reduces to

$$\lim_{x\to x^*} \|u_{\text{B}}(x, x^*) - u(x^*)\| = 0. \tag{58}$$

Applying (56), (58), and H3.6 to the triangle inequality in (57) we have

$$\lim_{x\to x^*} \|u(x) - u(x^*)\| = 0, \tag{59}$$

which completes the proof. ■

VI. Case Study: Analytic example

The following example shows that even relatively simple quadratic programming problems can have minimizers that are discontinuous with respect to a parameter of the problem. For instance consider the following:

$$\mathcal{P}(x) \quad = \quad \begin{cases} \text{argmin}_{u\in R} & \frac{1}{2}u^2 - ku \\ \text{subject to} & xu \le 0 \end{cases} \tag{60}$$

Note that for any x and k the nonlinear program $\mathcal{P}(x)$ is one-dimensional, strictly convex, analytic, and has a unique solution and is thus extremely well-posed. However, if $x = 0$ (regardless of the value of k) the linear independent constraint qualification does not hold. As a result many traditional smoothness results (such as Theorem 1) do not apply. In this case, the new results of Theorem 3 are useful in analyzing continuity (or the lack thereof). In the notation of Theorem 3, for $x^* = 0$,

$$\begin{array}{rcll} A_{\rm V}(x,x^*) & = & x & \\ b_{\rm V}(x,x^*) & = & 0 & \\ u_{\rm B}(x,x^*) & = & k & \\ u_{\rm S}(x,x^*) & = & 0 & \text{for } x \neq x^* \\ \Delta(x,x^*) & = & -k & \text{for } x \neq x^* \end{array} \quad (61)$$

Note that if $k = 0$ then conditions of Theorem 3 are all satisfied and the unique minimizer $u(x)$ is continuous at $x^* = 0$. However if $k \neq 0$ then H3.6 fails, and Theorem 3 no longer applies. The solution of $\mathcal{P}(x)$ can be found in closed-form

$$u(x) = \begin{cases} k & \text{if } kx \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad (62)$$

The above is discontiuous at $x = 0$ if $k \neq 0$ and is continuous at $x = 0$ if $k = 0$ (as predicted by Theorem 3).

VII. Case Study: Setpoint Regulation for the Compass Gait Walker

The main theorem of this paper (Theorem 3) will be illustrated in the design of a feedback controller for position regulation of a planar biped. The system of interest in this study is the compass gait biped shown in Figure 1. Specific mass and length parameters of the model are given in Table 1 of [23]. The equations of motion of this system are

$$M(q)\ddot{q} + N(q,\dot{q}) = u, \quad (63)$$

where $M(q)$ is the inertial matrix, $N(q,\dot{q})$ is a vector of Coriolis and gravity terms, and u is a vector of input torques. The state vector is $x = (q^T, \dot{q}^T)^T$. Note that in contrast to the analysis of [23] we will analyze the compass gait as a fully actuated system.

The control for this example regulates each of the joint angles, q_i, to a reference configuration, q_i^d. To this end, define the following virtual constraint: $y_i = q_i - q_i^d, \quad i \in 1,2$. As the reference configuration is static, the time derivatives of the outputs are simply the joint velocities $\dot{y}_i = \dot{q}_i, i \in 1,2$ and

$$\ddot{y}_i = L_f^2 y_i(x) + L_g L_f y_i(x) u, \quad i \in 1,2, \quad (64)$$

where for this simple control system $L_f^2 y(x) = -M^{-1}(x)N(x)$ and $L_g L_f y(x) = M^{-1}(x)$. Note that because the inertia matrix is positive definite and symmetric, it is invertible for all x.

A. Analyzing the CLF-QP for minimizing torques

The following feedback is given as the minimizer quadratic program over u, with a cost function that penalizes joint torques. This cost function reflects an intuitive choice in the control of robots as it is generally desirable to perform control using the least amount of torque

$$\mathcal{P}(x) = \begin{cases} \text{argmin}_{u \in R^n} \ \frac{1}{2} u^T u \\ \text{subject to} \quad A(x)u \leq b(x) \end{cases} \quad (65)$$

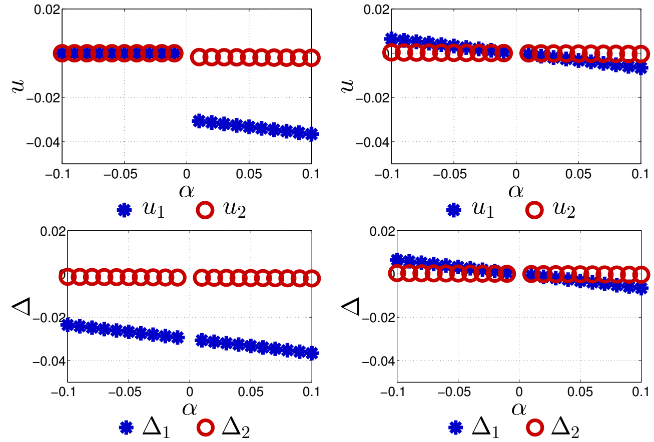

Fig. 2. **Left Column:** *The compass gait controller obtained by solving* (65) *along the set of test points* (70) *shows a jump discontinuity in the torques u at $\alpha = 0$; the corresponding $\lim_{x \to x^*} \Delta(x,x^*) \neq 0$ and thus, H3.6 of Theorem 3 is violated.* **Right Column:** *Setting gravity to zero in the compass gait simulation and solving* (65) *along the set of test points* (70) *results in continuous torques; the corresponding $\lim_{x \to x^*} \Delta(x,x^*) = 0$.*

with

$$A(x) = L_G V_\varepsilon(x) L_g L_f y(x) \quad (66)$$

$$b(x) = -\frac{c_3}{\varepsilon} V_\varepsilon(x) - L_F V_\varepsilon(x) - L_G V_\varepsilon(x) L_f^2 y(x). \quad (67)$$

Using the equations developed in Section IV, the above expands to

$$\begin{array}{rcl} A(x) & = & 2\eta(x)^T P_\varepsilon G M^{-1}(x) \\ b(x) & = & -\eta(x)^T (F^T P_\varepsilon + P_\varepsilon F + \frac{c_3}{\varepsilon} P_\varepsilon)\eta(x) \ \ldots \\ & & +2\eta(x)^T P_\varepsilon G M^{-1}(x) N(x) \end{array}$$

We will now check the conditions of Theorem 3 at a point $x^* = (q_1^d, q_2^d, 0, 0)^T$. In the notation of the main theorem

$$\begin{array}{l} A_{\rm V}(x,x^*) = A(x) \\ b_{\rm V}(x,x^*) = b(x) \\ u_{\rm B}(x,x^*) = 0 \\ u_{\rm S}(x,x^*) = A^T(x)(A(x)A^T(x))^{-1} b(x), \ \text{for } x \neq x^* \end{array}$$

Due to the relatively simple structure of the feedback (65), all assumptions of Theorem 3 are straightforward to verify with the exception of the last, H3.6. To test H3.6, choose a sequence $\{x_i\}_{i=0}^\infty$, for $x_i = (q_1^d, q_2^d, \alpha_i, \alpha_i)^T$. Choose $\alpha_i = 2^{-i}$, which leads to the properties that $M(x_i) = M(x_0)$, $\eta(x_i) = \alpha_i \eta(x_0)$, and $\lim_{i\to\infty} \alpha_i = 0$, so that

$$\Delta(x_i, x^*) = \alpha_i k_0(x_0) k_1(x_0) + k_0(x_0) k_2(x_0) N(x_i) \quad (68)$$

where

$$\begin{array}{l} k_0(x) = A^T(x)(A(x)A^T(x))^{-1} \\ k_1(x) = -\eta(x)^T (F^T P_\varepsilon + P_\varepsilon F + \frac{c_3}{\varepsilon} P_\varepsilon)\eta(x) \\ k_2(x) = 2\eta(x)^T P_\varepsilon G M^{-1}(x) \end{array} \quad (69)$$

Taking the limit of our sample sequence we have

$$\lim_{i\to\infty} \Delta(x_i, x^*) = k_0(x_0) k_2(x_0) N(x^*), \ \text{and} \ \lim_{i\to\infty} x_i = x^*$$

showing that H3.6 is violated if $k_0(x_0) k_2(x_0) N(x^*) \neq 0$.

B. Numeric Example

We are interested in the behavior of the controller (65) near a point of interest $x^* = (q_1^d, q_2^d, 0, 0)^T$. To help illustrate the singularities that arise, we will examine the torque requested by the feedback controller along a set of points

$$x = (q_1^d, q_2^d, 0, 0)^T + \alpha(0, 0, 1, 1)^T \tag{70}$$

where $-0.1 \leq \alpha \leq 0.1$. Note that controller experiences a jump discontinuity at $x = (q_1^d, q_2^d, 0, 0)$ if $N(q^d, \dot{q}^d) \neq 0$.

See Figure 2 for a simulation along the sequence described above. The desired configuration ($q_1^d = q_2^d = 0.1$) is not located at a kinematic singularity, there is no unusual interaction with the ground plane, and the decoupling matrix is not singular. Note that when H3.6 is violated, continuity is not expected, which is the case in the left column of Figure 2. If the same calculations are repeated, with the exception that $g = 0$ (that is, if we assume that the model (63) has no gravity term), then assumptions H3.1-H3.5 can be verified analytically, and H3.6 can be verified in simulation. Consistent with Theorem 3, the right column of Figure 2 shows that the controller is continuous at x^*.

VIII. Conclusion

Feedback controllers based on nonlinear programming problems seldom have closed form solutions. If we need to establish continuity or differentiability of the minimizer, then the controller can be analyzed either as a min-norm controller or more generally as a perturbed nonlinear programming problem. Under conditions provided in [9], Section 4.2, the min-norm controller is known to exist and to be continuous. In general however, if the perturbed nonlinear program (3) cannot be put into the form of the min-norm controller, or if a result stronger than continuity is needed, then alternative results such as Fiacco's theorem (Theorem 1) can be applied if the correct conditions are met.

This paper presents two novel contributions in Theorems 2 and 3. In Theorem 2 we apply Fiacco's theorem to show that for a RES-CLF controller with a single inequality constraint, as in (22), the resulting feedback is continuously differentiable at all points where $L_G V_\varepsilon \neq 0$. In Theorem 3 (the main theorem) we extend beyond the min-norm controller and Fiacco's theorem to state sufficient conditions for continuity of QP optimization-based control without assuming linear independence of the constraints. Most notably these conditions are relevant when $\eta = 0$ in a RES-CLF controller with one or more CLFs. The resulting continuity conclusions that are similar to the min-norm controller, but available to a wider variety of QP optimization-based controllers, including those with multiple active equality and inequality constraints.

The main theorem is illustrated in two examples. The first is a simple one-dimensional QP that demonstrates a continuity condition predicted by the main theorem (Section VI). The second is a simulation study of setpoint regulation in the two link-walker exhibiting continuity properties that are consistent with the main theorem (Section VII).

References

[1] "CLF based QP control on DURUS," Online: http://youtu.be/hYaUzE2lZN4.

[2] A. D. Ames and M. Powell, "Towards the unification of locomotion and manipulation through control lyapunov functions and quadratic programs," in *Control of Cyber-Physical Systems*. Springer, 2013, pp. 219–240.

[3] A. Ames, K. Galloway, K. Sreenath, and J. Grizzle, "Rapidly exponentially stabilizing control lyapunov functions and hybrid zero dynamics," *Automatic Control, IEEE Transactions on*, vol. 59, no. 4, pp. 876–891, April 2014.

[4] J. F. Bonnans and A. Shapiro, *Perturbation analysis of optimization problems*. Springer Science & Business Media, 2000.

[5] E. Cousineau and A. D. Ames, "Realizing underactuated bipedal walking with torque controllers via the ideal model resolved motion method," in *IEEE International Conference on Robotics and Automation*, submitted, 2015.

[6] A. V. Fiacco and Y. Ishizuka, "Sensitivity and stability analysis for nonlinear programming," *Annals of Operations Research*, vol. 27, pp. 215–236, 1990.

[7] A. V. Fiacco, "Sensitivity analysis for nonlinear programming using penalty methods," *Mathematical programming*, vol. 10, no. 1, pp. 287–311, 1976.

[8] A. V. Fiacco and G. P. McCormick, *Nonlinear programming: sequential unconstrained minimization techniques*. Siam, 1969, vol. 4.

[9] R. A. Freeman and P. V. Kokotović, *Robust Nonlinear Control Design*. Birkhäuser, 1996.

[10] K. Galloway, K. Sreenath, A. D. Ames, and J. W. Grizzle, "Torque saturation in bipedal robotic walking through control lyapunov function based quadratic programs," to appear in IEEE Access, 2015.

[11] K. Jittorntrum, "Solution point differentiability without strict complementarity in nonlinear programming," in *Sensitivity, Stability and Parametric Analysis*. Springer, 1984, pp. 127–138.

[12] S. Kuindersma, F. Permenter, and R. Tedrake, "An efficiently solvable quadratic program for stabilizing dynamic locomotion," *CoRR*, 2013.

[13] M. J. Kurtz and M. A. Henson, "Input-output linearizing control of constrained nonlinear processes," *Journal of Process Control*, 1997.

[14] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Survey constrained model predictive control: Stability and optimality," *Automatica*, 2000.

[15] B. Morris, M. J. Powell, and A. D. Ames, "Sufficient conditions for the lipschitz continuity of qp-based multi-objective control of humanoid robots," in *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*. IEEE, 2013, pp. 2920–2926.

[16] R. M. Murray, J. Hauser, A. Jadbabaie, M. B. Milam, N. Petit, W. B. Dunbar, and R. Franz, "Online control customization via optimization-based control," in *In Software-Enabled Control: Information Technology for Dynamical Systems*. Wiley-Interscience, 2002.

[17] M. J. Powell, E. Cousineau, and A. D. Ames, "Model predictive control of underactuated bipedal robotic walking," in *IEEE International Conference on Robotics and Automation*, to appear, 2015.

[18] S. M. Robinson, *Generalized equations and their solutions, Part II: Applications to nonlinear programming*. Springer, 1982.

[19] M. Spong, J. Thorp, and J. Kleinwaks, "The control of robot manipulators with bounded input," *IEEE Transactions on Automatic Control*, vol. AC-31, no. 6, pp. 483–490, 1986.

[20] B. Stephens and C. Atkeson, "Push recovery by stepping for humanoid robots with force controlled joints," in *International Conference on Humanoid Robots*, Nashville, Tennessee, 2010.

[21] T. Sugihara and Y. Nakamura, "Whole-body coooperative balancing of humanoid robot using COG jacobian," *Proc. IEEE/RSJ Int. Conf. Intell. Robots and Systems*, vol. 3, pp. 2575–2580, 2002.

[22] Y. Tassa, N. Mansard, and E. Todorov, "Control-limited differential dynamic programming," in *IEEE Conference on Robotics and Automation (ICRA)*, 2014.

[23] E. Westervelt, B. Morris, and K. Farrell, "Analysis results and tools for the control of planar bipedal gaits using hybrid zero dynamics," *Autonomous Robots*, vol. 23, no. 2, pp. 131–145, 2007.

[24] P.-B. Wieber, "Trajectory free linear model predictive control for stable walking in the presence of strong perturbations," *Proc. 6th IEEE-RAS Intl. Conf. on Humanoid Robots*, pp. 137–142, 2006.

[25] H. Zhao, J. Horn, J. Reher, V. Paredes, and A. D. Ames, "Realization of nonlinear real-time optimization based controllers on self-contained trafemoral prosthesis," in *International Conference on Cyber-Physical Systems (ICCPS)*, to appear, 2015.