Energy Shaping of Hybrid Systems via Control Lyapunov Functions

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***Abstract*— This paper presents a method for adding robustness to periodic orbits in hybrid dynamical systems by shaping the energy. This energy shaping method is similar to existing methods but improves upon them by utilizing control Lyapunov functions which allow for formal results on stability. The main theoretical result, Theorem 1, states that, given an exponentially-stable limit cycle in a hybrid dynamical system, application of the presented energy shaping controller results in a closed-loop system which is exponentially stable. As illustrated through simulation on the compass gait biped, this method turns out to be useful in practice, providing an improvement in robustness and convergence properties while maintaining existing stability.**

I. Introduction

Passive dynamic walking and passivity-based control methods have been studied for arguably longer than most other locomotion control strategies. Perhaps the most attractive feature of these methods stems from the readily observable intuition underlying their design. In the late 1980's McGeer described experiments which demonstrated that planar, passive bipeds could walk stably down shallow slopes for designs like the compass gait biped [1] and the kneed compass gait biped [2]. This work gave rise to the term *passive dynamic walking* and spurred the creation of additional robots with similar design principles as described in, e.g., [3], whose operational procedures involved injecting small amounts of energy to achieve passivity-based gaits.

This paper examines passive gaits which, in general, have restrictive stability properties and proposes a method for improving the robustness of these gaits through an understanding of energy. The method presented, styled *energy shaping*, is similar in concept to the method of total energy shaping as presented in [4], which acts to shape the energy of the system but does so in a way which only guarantees asymptotic stability with respect to an arbitrary energy level and does not guarantee exponential stability of the overall gait and may in fact destabilize gaits. The method of energy shaping presented herein, in contrast, will stabilize the energy dynamics of a hybrid periodic orbit while maintaining stability of the hybrid system.

Numerous methods currently exist for gait design but, aside from specific methods which construct a zero dynamics such as human-inspired control [5], many of these methods do not have an intrinsic concept of stabilization to a specific gait through gain adjustment. This is especially true of passivity-based methods such as controlled symmetries [6] and other controlled Lagrangian methods [7]. Energy shaping owes its development to the observation that the total energy of a system is conserved in the absence of non-conservative forcing. For hybrid systems – systems which combine continuous dynamics such as leg swing with discrete dynamics such as foot strike – the conservation of energy through the continuous dynamics means that the change in energy level occurs from the discrete events in the system – foot-strike for bipeds – which exert non-conservative impulsive forcing. By adding control to the continuous dynamics, overall stability properties of a gait tend to improve as has been observed in, e.g., [8], and as will be demonstrated later in this paper through simulation. Formulation of the control objective using a control Lyapunov function makes it possible to achieve these improvements while simultaneously guaranteeing the existence of a control law which does not destabilize the system.

The rest of the paper is structured as follows: Section II provides background information on hybrid systems which are used for modeling in this paper. Section III presents the main theoretical results of this paper in the form of Theorem 1. Section IV shows how to reformulate a hybrid system through a coordinate transformation that is later used to prove the results of Theorem 1 in Section V. In Section VI, simulation results demonstrating the effectiveness of energy shaping are outlined. Finally, Section VII reflects on the ideas presented in this paper and poses questions for future study.

II. Hybrid Systems

Consider a hybrid dynamical system (cf. [9]) with total energy $E(\chi)$. The system has coordinates $\chi \in D$ which take values in the *domain of admissibility*,

$$D = \{\chi \times \mathbb{R}^n : h(\chi) \geq 0\},$$

where the discrete aspect comes from a *guard constraint*, $h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, which leads to a transverse plane, $S \subset \mathbb{R}^n$, called the *switching surface*,

$$S = \left\{\chi \in \mathbb{R}^n : h(\chi) = 0 \text{ and } \dot{h}(\chi) < 0\right\}. \tag{1}$$

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The uncontrolled hybrid system can be written

$$\Sigma = \begin{cases} \dot{\chi} = F(\chi), & \chi \in D \setminus S, \\ \chi^+ = \Delta(\chi^-), & \chi \in S, \end{cases} \tag{2}$$

where F is a smooth vector field and Δ is a smooth map called the reset map (see, e.g., [10]). Under the action of control effort u, the hybrid control system has the form

$$\Sigma_c = \begin{cases} \dot{\chi} = F(\chi) + G(\chi)\,u, & \chi \in D \setminus S, \\ \chi^+ = \Delta(\chi^-), & \chi \in S, \end{cases} \tag{3}$$

for control values $\mathscr{U} \subseteq \mathbb{R}^m$ and smooth vector field G.

Solutions to hybrid systems are defined in the traditional manner; see, e.g., [5]. For an initial condition $\chi_0 \in D$, the solution $x(t)$ evolves according to the continuous dynamics in (2) until it reaches the edge of D, intersecting the guard S by assumption (tranversality). Then the reset map Δ is applied and the solution is again governed by the continuous dynamics. Stability is defined in terms of the *ω limit set*. Let $\phi_t(\chi)$ represent the flow of the continuous dynamics. A set $\mathscr{O}$ is a *periodic orbit* with period $T > 0$ if

$$\mathscr{O} = \{\chi(t) \in D : \chi(0) = \phi_T(\Delta(\chi(0)))\}.$$

III. Energy Shaping

The proposed controller relies on control Lyapunov functions [11], [12]. To guarantee a stricter form of convergence, a modified notion of these functions is used:

Definition 1: For the continuous dynamics of (3)*, a continuously differentiable function $V_\varepsilon : D \to \mathbb{R}_{\geq 0}$ is said to be a **rapidly exponentially stablizing control Lyapunov function (RES–CLF)** (see [13]) if there exist constants $c_1, c_2, c_3, c_4 \in \mathbb{R}_{>0}$ such that for all $\varepsilon > 0$ and for all $\chi \in D$,*

$$c_1\|\chi\| \leq V_\varepsilon(\chi) \leq \tfrac{c_2}{\varepsilon^2}\|\chi\|,$$
$$\inf_{u\in\mathscr{U}} \left[L_\Phi V_\varepsilon(\chi) + L_\Gamma V_\varepsilon(\chi)\,u + \tfrac{c_3}{\varepsilon}V_\varepsilon(\chi)\right] \leq 0.$$

where $L_f h(\chi) = \frac{\partial h(\chi)}{\partial \chi} \cdot f(\chi)$ is the Lie derivative representing the flow of $h(\chi)$ along the vector field $\dot{\chi} = f(\chi)$.

For the continuous dynamics, define a candidate Lyapunov function, $V : D \to \mathbb{R}_{\geq 0}$, of the form

$$V(\chi) = \frac{1}{2}(E(\chi) - E_{\text{ref}})^2, \tag{4}$$

with E_{ref} the energy associated with the periodic orbit, and use it to construct the energy shaping controller

$$\begin{aligned} \mu_\varepsilon(\chi) = \underset{u(\chi)\in\mathbb{R}^m}{\arg\min}\ & u(\chi)^T u(\chi) \\ \text{s.t. } & L_F V(\chi) + L_G V(\chi)\,u(\chi) + \frac{c_3}{\varepsilon}V(\chi) \leq 0. \end{aligned} \tag{5}$$

Applying this to the system (3) results in

$$\Sigma = \begin{cases} \dot{\chi} = F(\chi) + G(\chi)\,\mu_\varepsilon(\chi), & \chi \in D \setminus S, \\ \chi^+ = \Delta(\chi^-), & \chi \in S. \end{cases} \tag{6}$$

As a result of the control law construction in (5), the closed-loop dynamics of (6) is stabilized with respect to the zero level set of the Lyapunov function (4) thus satisfying the convergence guarantee specified in Definition 1.

With the preceding setup in mind, the main formal idea behind energy shaping can now be stated:

Theorem 1: Given an exponentially-stable limit cycle in a hybrid system of the form (2)*, application of the energy shaping controller* (5) *to the control system* (3) *results in the closed-loop hybrid system* (6)*, which is exponentially stable.*

A sketch of the proof is given later after some discussion.

IV. Zero Dynamics Formulation

In order to understand the nature of energy shaping, consider breaking up the system into two sets of coordinates,

$$\dot{x} = f(x,z) + g(x,z)\,u, \qquad \dot{z} = q(x,z) + w(x,z)\,u, \tag{7}$$

with states $x \in X$ and $z \in Z$ and control inputs $u \in \mathscr{U}$. The vector fields f, g, q, and w are assumed to be locally Lipschitz continuous. To simplify notation, define

$$\Phi(x,z) = (f(x,z),\, q(x,z)), \quad \Gamma(x,z) = (g(x,z),\, w(x,z)).$$

The natural choice of transformation to convert the continuous dynamics of (3) to (7) is through energy. Thus, for mechanical systems where $\chi = (q, \dot{q}) \in T\mathscr{Q}$, consider the transformation

$$x = E(\chi) - E_{\text{ref}}, \quad z = (q_1,\ \ldots,\ q_n,\ \dot{q}_1,\ \ldots,\ \dot{q}_{n-1}),$$

where n is the size of the configuration space, $\mathscr{Q}$. By construction, the fixed point of the hybrid system can be chosen to occur at $(x,z) = (0, z^*)$ such that $\Delta(0, z^*) = (0, \mathbf{0})$. Moreover, because energy does not change by the natural dynamics, $f(x,z) = 0$. The transformation is valid if it is locally diffeomorphic. By examining the determinant of the transformation, it becomes clear that the transformation is valid almost everywhere.

A. Exponential Stability

Applying the control law (5), the dynamics (7) becomes

$$\begin{aligned} \dot{x} &= f(x,z) + g(x,z)\,\mu_\varepsilon(x,z), \\ \dot{z} &= q(x,z) + w(x,z)\,\mu_\varepsilon(x,z). \end{aligned}$$

By the construction of the control law (5), it is clear that $\mu_\varepsilon(0,z) = 0$ and thus it follows that $f(0,z) = 0$. In other words, the zero dynamics manifold $\mathscr{Z}$ is the restricted subset of X such that $x = 0$. Rewrite (4) in the zero dynamics coordinates, $V(x) = \frac{1}{2}x^2$, and consider the following:

Proposition 1: Exponential stability of the continuous x dynamics is guaranteed if a RES–CLF exists satisfying

$$
\begin{gathered}
c_1 |x|^2 \leq V(x)) \leq \tfrac{c_2}{\varepsilon^2} |x|^2 , \\
\inf_{u \in \mathscr{U}} \left[L_\Phi V(x,z) + L_\Gamma V(x,z)\, u + \tfrac{c_3}{\varepsilon} V(x) \right] \leq 0,
\end{gathered} \tag{8}
$$

for all $(x,z) \in X \times Z$.

Proof (Sketch): It is easy to see that the first inequality is satisfied for $c_1 \leq \frac{1}{2}$ and $c_2 \geq \frac{\varepsilon^2}{2}$. Define the set

$$K_\varepsilon = \left\{ u \in \mathscr{U} : L_\Phi V(x,z) + L_\Gamma V(x,z)\, u + \frac{c_3}{\varepsilon} V(x) \leq 0 \right\}.$$

Examining this set in the context of mechanical systems results in a locally Lipschitz solution for fully-actuated systems. It follows that solutions satisfy

$$\|x(t)\| \leq \frac{1}{\varepsilon} \sqrt{\frac{c_2}{c_1}} e^{-\frac{c_3}{2\varepsilon} t} \|x(0)\|. \qquad \blacksquare$$

Combine (7) with the reset map to obtain the hybrid control system $\overline{\Sigma}_c$ (as in (3)). Applying a Lipschitz continuous control law (which takes values in K_ε) results in

$$
\overline{\Sigma}_\varepsilon = \begin{cases}
\begin{array}{rcl} \dot{x} &=& f(x,z) + g(x,z)\, \mu_\varepsilon(x,z) \\ \dot{z} &=& q(x,z) + w(x,z)\, \mu_\varepsilon(x,z) \end{array} & \text{if } (x,z) \in D \setminus S, \\
\begin{array}{rcl} x^+ &=& \Delta_X(x^-, z^-) \\ z^+ &=& \Delta_Z(x^-, z^-) \end{array} & \text{if } (x,z) \in S.
\end{cases} \tag{9}
$$

V. Proof of Main Result

In order to achieve the stated goal, it is necessary to show that, given a system with a limit cycle representing the desired behavior, energy shaping can be applied and the resulting system will have an invariant orbit which is equivalent to the nominal system. Simply put, the control contribution from the energy shaping controller must be identically zero on the orbit. Consider the following lemma:

Lemma 1: The shaped system (6) *demonstrates a periodic orbit which is identical to the unshaped system* (2).

Proof: For states on the periodic orbit, i.e., $\chi^{\mathscr{O}} \in \mathscr{O}$, the energy is a known constant, $E(\chi^{\mathscr{O}}) = E_{\text{ref}}$. Therefore, the limit cycle represents an invariant level set of the energy. By construction of the Lyapunov function (4) used in the controller (5), it is clear that $V(\chi^{\mathscr{O}}) = 0$ and, moreover, that $\inf_{\chi \in D} V(\chi) = 0$. The solution to the optimization problem (5) has cost $u(\chi)^T u(\chi) = 0$ (which implies that all elements of $u(\chi)$ are zero when $V(\chi) = 0$) and this satisfies the stability condition of the control Lyapunov function; indeed $\dot{V}(\chi^*) = 0$ since the energy does not change without external forcing. Thus, the periodic orbits are equivalent. $\blacksquare$

Let the Poincaré map of (6) be denoted $P_\varepsilon : S \to S$ and let $\phi_t(x,z)$ represent a flow of the vector field for time t starting from state (x,z). The Poincaré map takes the form $P_\varepsilon(x,z) = \phi^\varepsilon_{T_I^\varepsilon(x,z)}(\Delta(x,z))$, where $T_I^\varepsilon(x,z)$ is the time to impact. Before proving the main theorem, some bounds related to the Poincaré map must be established using arguments similar to those presented in [13], [10]. Note that the reset map is locally Lipschitz continuous about the fixed point $(x,z) = (0,z^*)$ and $\Delta_X(0,z^*) = 0$ thus

$$\|\Delta_X(x,z) - \Delta_X(0,z^*)\| \leq L_\Delta \|(x, z - z^*)\| \tag{10}$$

for some $(x,z) \in B_\gamma(0,z^*)$ with $\gamma > 0$ where L_Δ is the Lipschitz constant of $\Delta(x,z)$. Now, consider the following bounds on the time-to-impact functions and Poincaré maps:

Lemma 2: For the control system (9)*,*

$$
\begin{gathered}
|T_I^\varepsilon(\Delta(x,z)) - T_I(\Delta(x,z))| \leq A_{T_I}(\varepsilon) \|(x, z - z^*)\|, \\
\|P_\varepsilon(x,z) - P(x,z)\| \leq A_P(\varepsilon) \|(x, z - z^*)\|,
\end{gathered}
$$

where $\lim_{\varepsilon \nearrow +\infty} A_{T_I}(\varepsilon) = 0$ *and* $\lim_{\varepsilon \nearrow +\infty} A_P(\varepsilon) = 0$.

Proof (Sketch): Consider the Poincaré section $\mathscr{P}$ which is the guard from (1). Using the change of coordinates $\rho(\varepsilon) := \frac{1}{\varepsilon}$, define the function

$$N(t, \rho, x, z) = h(\phi_t^\rho(\Delta(x,z))),$$

which is locally Lipschitz continuous in x, z, and ρ by construction as a composition of Lipschitz continuous functions. By the transversality assumption, it follows that

$$\frac{\partial N(T, 0, 0, z^*)}{\partial t} = \dot{h}(\phi_t^0(\Delta(0,z^*)) \neq 0.$$

By the implicit function theorem [14], one can show that there is a time-to-impact function $T_I^\varepsilon(x,z)$ satisfying

$$0.9 T^* \leq T_I^\varepsilon(x,z) \leq 1.1 T^*.$$

The rest of the proof involves constructing an auxiliary time-to-impact function that is locally Lipschitz continuous and independent of ε and then relating it to T_I^ε in a similar way as was done in [13]. Next, μ must be bounded. The explicit solution to the QP (5) is given by the min-norm control law [11]:

$$\mu_\varepsilon(x,z) = -\frac{\psi_0(x,z)\,\psi_1(x,z)}{\psi_1^T(x,z)\,\psi_1(x,z)},$$

with

$$\psi_0(x,z) := L_\Phi V_\varepsilon(x,z) + \frac{c_3}{\varepsilon}, \quad \psi_1(x,z) := \left(L_\Gamma V_\varepsilon(x,z)\right)^T,$$

where $L_\Phi V_\varepsilon(x,z) \equiv 0$ by the choice of $V_\varepsilon(x,z)$ as energy. Since energy does not change by the natural dynamics,

$$\mu_\varepsilon(x,z) = -\frac{\frac{c_3}{\varepsilon} V_\varepsilon(x,z) \Gamma^T(x,z) \left(\frac{\partial V_e(x,z)}{\partial(x,z)}\right)^T}{\frac{\partial V_e(x,z)}{\partial(x,z)} \Gamma(x,z) \Gamma^T(x,z) \left(\frac{\partial V_e(x,z)}{\partial(x,z)}\right)^T}.$$

If Γ is full rank and takes values in a compact set for states near the orbit, i.e., $\Gamma(x,z) \in \Omega$ and $d_1 \leq \|\Gamma(x,z)\| \leq d_2$, then

$$\|\mu_\varepsilon(x,z)\| \leq \frac{c_2 c_3}{\varepsilon^3} \frac{\lambda_{\max}}{\lambda_{\min}} |x|, \tag{11}$$

with

$$\lambda_{\max} := \sup\{\lambda_{\max}\Gamma(x,z) : (x,z) \in \Omega\},$$
$$\lambda_{\min} := \inf\{\lambda_{\min}\Gamma(x,z) : (x,z) \in \Omega\},$$

where $\Omega \subset D$ is a stable tube around the periodic orbit $\mathcal{O}$. In addition, by the Lipschitz assumption Φ is bounded around $\mathcal{O}$, hence $\Upsilon := \sup\{\|\Phi(x,z)\| : (x,z) \in \Omega\}$. Using (11) and Proposition 1 along with the auxiliary time-to-impact function allows one to establish a bound of the form

$$|T_I^\varepsilon(x,z) - T_I(x,z)| \leq L_B\, \theta(\varepsilon)\, \|(x, z - z^*)\|.$$

Defining $A_{T_I}(\varepsilon) := L_B\, \theta(\varepsilon)$ establishes the first part of the lemma. The proof is completed by showing that

$$\|P_\varepsilon(x,z) - P(x,z)\| \leq A_P(\varepsilon) \|(x, z - z^*)\|$$

with $A_P(\varepsilon) := (1 + L_B)\, \theta(\varepsilon)\, \|(x, z - z^*)\|$. In establishing the bounds, one can observe that limiting behavior is $\lim_{\varepsilon \nearrow +\infty} A_P(\varepsilon) = 0$. ∎

Now Theorem 1 can be proven:

Proof (Sketch): [Theorem 1] By the discrete converse Lyapunov theorem, exponential stability of $\mathcal{O}$ implies the existence of a discrete Lyapunov function $V_n : B_\delta(0, z^*) \cap S \to \mathbb{R}_{\geq 0}$ satisfying

$$\begin{aligned} &r_1 \|(x,z)\|^2 \leq V_n(x,z) \leq r_2 \|(x, z - z^*)\|^2, \\ &V_n(P(x,z)) - V_n(x,z) \leq -r_3 \|(x, z - z^*)\|^2, \\ &|V_n(x,z) - V_n(x',z')| \leq r_4 \|(x, z - z^*) - (x', z' - z^*)\| \cdot \\ &\qquad (\|(x, z - z^*)\| + \|(x', z' - z^*)\|) \end{aligned} \tag{12}$$

for some $r_1, r_2, r_3, r_4 \in \mathbb{R}_{>0}$. In addition, consider the CLF associated with (5) which is $V_\varepsilon : X \to \mathbb{R}_{\geq 0}$. Denote by $V_{\varepsilon,X} = V_\varepsilon|_S$ the restriction of the CLF V_ε to the switching surface S. Using these Lyapunov functions, define the candidate Lyapunov function

$$V_{P_\varepsilon}(x,z) = V_n(x,z) + \sigma V_{\varepsilon,X}(x).$$

From (8) and (12), it is apparent that $V_{P_\varepsilon}(x,z)$ is bounded:

$$\begin{aligned} &\sigma c_1 |x|^2 + r_1 \|(x, z - z^*)\|^2 \\ &\qquad \leq V_{P_\varepsilon}(x,z) \leq \sigma \frac{c_2}{\varepsilon^2} |x|^2 + r_2 \|(x, z - z^*)\|^2. \end{aligned}$$

Next, note that

$$\begin{aligned} &V_{P_\varepsilon}(P_\varepsilon(x,z)) - V_{P_\varepsilon}(x,z) = \\ &\quad V_n(P_\varepsilon(x,z)) - V_n(x,z) + \sigma(V_{\varepsilon,X}(P_\varepsilon(x,z)) - V_{\varepsilon,X}(x)). \end{aligned} \tag{13}$$

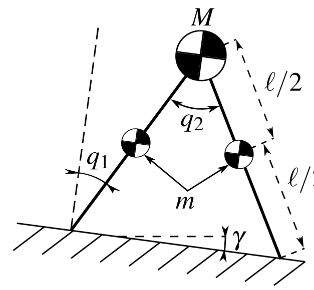

Fig. 1: Compass-gait biped with walking down a slope.

By construction of the control law (5) and as a consequence of (10) and Lemma 2, one can establish that

$$V_{\varepsilon,X}(P_\varepsilon^x(x,z)) - V_{\varepsilon,X}(x) \leq \beta_1(\varepsilon) \|(x, z - z^*)\|^2 - \frac{c_2}{\varepsilon^2} |x|^2$$

with $\beta_1(\varepsilon) := \frac{c_2}{\varepsilon^2} L_\Delta^2 e^{-\frac{c_3}{\varepsilon} .9T^*}$. Now the Lyapunov function guaranteed by the converse theorem must be bounded. As a result of Lemma 2 and the assumption of exponential stability about the origin, it follows that

$$\begin{aligned} &\|P_\varepsilon(x,z) - P(x,z)\| \leq A_P(\varepsilon) \|(x, z - z^*)\|, \\ &\|P_\varepsilon(x,z)\| \leq A_P(\varepsilon) \|(x, z - z^*)\| + L_P \|(x, z - z^*)\|, \\ &\|P(x,z)\| \leq N\alpha \|(x, z - z^*)\|, \end{aligned} \tag{14}$$

where L_P is the Lipschitz constant for P. Using (12) and (14) with (13) allows one to establish that

$$V_n(P_\varepsilon(x,z)) - V_n(x,z) \leq (\beta_2(\varepsilon) - r_3) \|(x, z - z^*)\|^2.$$

where, for simplicity, $\beta_2(\varepsilon) := r_4 A_P(\varepsilon)(N\alpha + A_P(\varepsilon) + L_P)$. The above bounds can be used to establish

$$\begin{aligned} &V_{P_\varepsilon}(P_\varepsilon(x,z)) - V_{P_\varepsilon}(x,z) \leq \\ &\qquad -\begin{pmatrix} |x| & \|z - z^*\| \end{pmatrix} \Lambda(\varepsilon) \begin{pmatrix} |x| \\ \|z - z^*\| \end{pmatrix} \end{aligned}$$

with

$$\Lambda(\varepsilon) = \begin{pmatrix} r_3 - \beta_2(\varepsilon) - \sigma(\beta_1(\varepsilon) - \frac{c_2}{\varepsilon^2}) & r_3 - \beta_2(\varepsilon) - \sigma\beta_1(\varepsilon) \\ r_3 - \beta_2(\varepsilon) - \sigma\beta_1(\varepsilon) & r_3 - \beta_2(\varepsilon) - \sigma\beta_1(\varepsilon) \end{pmatrix}.$$

By examining the determinant of $\Lambda(\varepsilon)$, one can ascertain that stability is achieved when $\beta_2(\varepsilon) + \sigma\beta_1(\varepsilon) < r_3$. Examining the limits, it becomes apparent that $\lim_{\varepsilon \nearrow +\infty} \beta_1(\varepsilon) = 0$ and $\lim_{\varepsilon \nearrow +\infty} \beta_2(\varepsilon) = 0$, and thus for small enough values of $\sigma > 0$ and large enough values of ε, stability is maintained. ∎

A more complete proof can be found in [15].

VI. Simulation Results

Due to the analytical complexities of bipedal walkers, the effectiveness of novel controllers is frequently demonstrated through numerical simulation. Using the compass gait biped

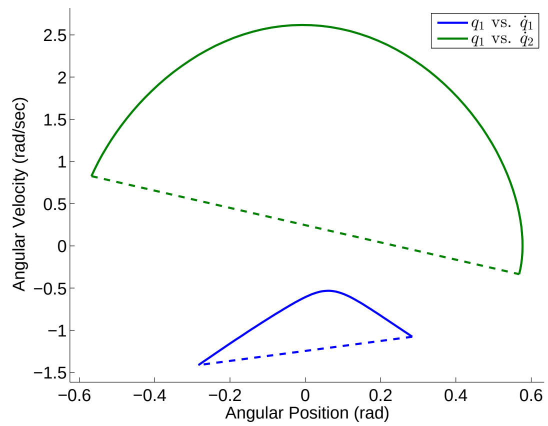

Fig. 2: Limit cycle of the passive compass gait biped.

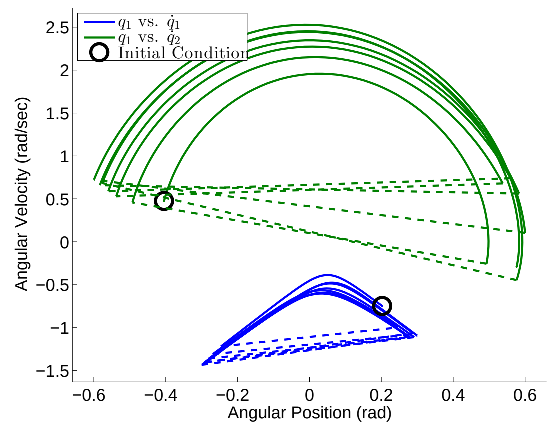

Fig. 4: Energy shaping allows recovery from distant states.

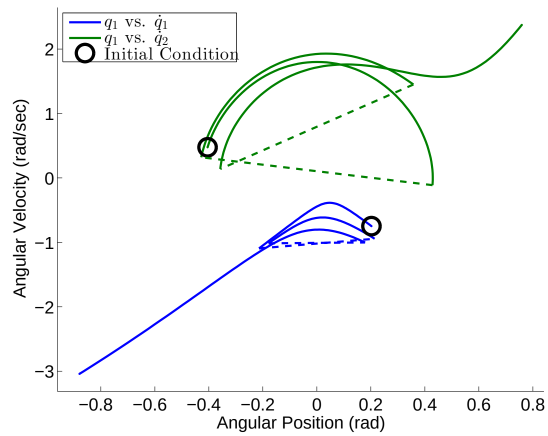

Fig. 3: The passive system cannot recover from distant states.

shown in Fig. 1 with model parameters $M = 20\ kg$, $m = 5\ kg$, $\ell = 1\ m$, and $\gamma = .05\ rads$, simulations were conducted to demonstrate the effectiveness of the energy shaping procedure. The limit cycle of the passive system is shown in Fig. 2 with the dotted lines representing discrete jumps from foot-strike (and coordinate relabeling). This gait has a fixed point

$$(q^*,\ \dot{q}^*) = (-0.2891,\ 0.5781,\ -1.4006,\ -0.2802) \quad (15)$$

on the guard with eigenvalues $|\lambda| = (0.5147, 0.5147, 0.0980)$ corresponding to a linearization of the Poincaré map restricted to the guard. Because these eigenvalues have magnitudes below unity, the corresponding hybrid periodic is locally exponentially stable. The impact map can be applied to the fixed point (15) to compute the post-impact coordinates

$$\Delta(q^*,\ \dot{q}^*) = (0.2891,\ -0.5781,\ -1.0681,\ 0.6797). \quad (16)$$

To see the benefits of energy shaping, consider two simulations conducted from a perturbed post-impact state,

$$(q_0,\ \dot{q}_0) = (0.2023,\ -0.4047,\ -0.7477,\ 0.4758), \quad (17)$$

which was naively obtained by multiplying the post-impact fixed point (16) by 0.7 resulting in a relatively large perturbation. For the passive walker, one can see from Fig. 3 that the biped falls on the third step. Using energy shapping with $\frac{c_3}{\varepsilon} = 1$ (Fig. 4), the biped is able to recover from the initial condition and quickly converges to the limit cycle.

In addition to the ostensible increase in robustness, energy shaping seems to improve convergence. To see this, a simulation was conducted from the starting point

$$(q_0,\ \dot{q}_0) = (0.2457,\ -0.4914,\ -0.9079,\ 0.5777),$$

which, in similarity to the previous simulation, was obtained by multiplying (16) by 0.85. This point had to be closer than (17) in order to fall within the domain of attraction (DOA) of the passive biped. The difference in convergence for the energy levels of the passive and shaped systems is shown in Fig. 5. One can see that the shaped system converges more quickly than the passive system. Whereas the passive system changes energy only through impact, the shaped system also converges during the continuous dynamics. Finally, stability is maintained for large enough values of ε but smaller values give better convergence so a trade-off naturally arises.

More comprehensive evidence for the expansion of the domain of attraction can be seen in Fig. 6 which provides a comparison of the stable region on the guard for both the passive and shaped systems. Because the guard is a tranverse hyperplane of the domain, it is a codimension-one submanifold. In Fig. 6, coordinates $(q_1, \dot{q}_1, \dot{q}_2)$ were chosen to parameterize the guard; q_1 is determined by the constraint that both feet are on the ground. It is interesting to note that the domain of attraction expands most readily into the region of low energy (small steps, small angular velocities) for which states the passive biped would simply lack the energy necessary to fall into a gait.

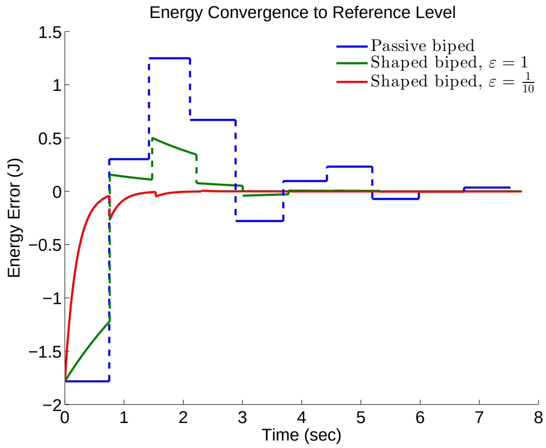

Fig. 5: Convergence has more desirable behavior with energy shaping. Smaller values of ε offer better convergence.

VII. Concluding Remarks

By its nature, local exponential stability does not necessarily provide much insight into stability of large regions around the limit cycle. Yet this type of information can be invaluable when investigating the practicality of a control law. Often, when control theorists talk about the robustness of controllers, there is some implication that this refers to stability properties over a larger range of operating states, or, in other words, the domain of attraction. It is notoriously difficult to compute the DOA even for a simple system – the simplest bipedal model, the compass gait biped, does not even have a closed form solution – and researchers often perform repeated simulations to create a visual representation of the DOA and analyze robustness properties in this respect. Thus it should be seen as unremarkable that claims about increasing the DOA of a limit cycle generally remain unsubstantiated from a formal theoretical standpoint. However, numerical simulations as in this paper can provide strong evidence for the effectiveness of a controller. Despite the fact that the energy shaping controller only guarantees local (exponential) stability, the simulations presented demonstrate that the region of stability is relatively large.

In addition to the expansion of the DOA, energy shaping also seems to provide more desirable convergence properties. Hence, when considered alongside the formal notion from Theorem 1 that energy shaping does not destabilize a system for proper gains, the numerical evidence presented provides a persuasive argument for the practicality and utility of energy shaping. As a final note, the formulation in this paper considers passive systems, yet the method seems to work with non-passive systems. But, a formal guarantee of stability is seemingly more complex than the proof for conservative systems and is, therefore, left as a question for future study.

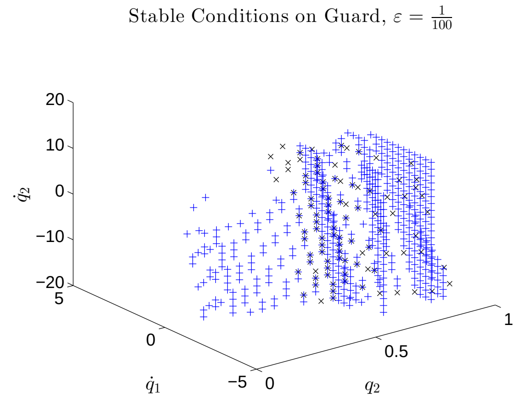

Fig. 6: The domain of attraction restricted to the guard for both the shaped (blue '+') and passive systems (black '×').

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