Hierarchical Control of Series Elastic Actuators through Control Lyapunov Functions

Matthew J. Powell and Aaron D. Ames

Abstract— **This paper addresses the control design problem of stabilizing both joint-angle and spring-torque control objectives in robots with series elastic actuators (SEAs). The proposed method is a hierarchical control scheme which employs rapidly exponentially stabilizing control Lyapunov functions (RES-CLFs) to obtain controllers for each tier in the scheme. In the main result of the paper, it is shown that for proper choice of controller parameters, applying the proposed controller to the SEA control system results in simultaneous exponential stability of the joint-angle and the spring-torque control objectives. Furthermore, it is shown that for a locally exponentially stable periodic orbit in the zero dynamics of the control system considered, the proposed controller renders a corresponding orbit in the full SEA dynamics locally exponentially stable.**

I. INTRODUCTION

Series elastic actuators are a frequent choice of mechanical drive in contemporary force-controlled robot manipulator applications, such as robots that work closely with other robots [5] and robots that work in close proximity to humans [18], [6]. By construction, SEAs are more amenable to accurate force control than alternative actuators as the elastic element in a SEA provides a direct force estimate [16]. In addition, some researchers leverage the passive dynamics of series elastic actuators to improve energy efficiency [10]. Per the force-control application of series elastic actuators, the corresponding control approaches of some researchers focus on high performance "inner loops" to stabilize actuator force [19], [18], sometimes through disturbance compensation [8], [14], while others prioritize obtaining desired actuator impedances through "passivity" methods [19], [13]. However, accurate force control comes with a trade-off: series elastic actuators suffer from lower zero force bandwidth [15] and increased holistic control design complexity due to the higher dimensional system and the increased degree of under-actuation [11].

This paper targets series-elastic actuator control design challenges in which achieving desired dynamics in the "post-spring" or "joint-angle" configuration coordinates is equally as important as stabilizing the actuator forces. An example of one such system is the control of stable periodic walking in a biped robot driven by series elastic actuators, in which the stability of joint-angle dynamics plays a much larger role in the stability of the entire robot than is the case in conventional manipulators. A large body of work leveraging the *hybrid zero dynamics* control design [20] and the *human-inspired control* [1] paradigms yields formal methods for achieving provably stable bipedal robotic walking for direct-drive robots through the establishment of *hybrid invariant* surfaces defined on joint-angle coordinates. In both methods, the formal results follow from the realization of controllers which exponentially stabilize functions of the joint-angle coordinates. However, achieving exponential stability in the joint-angle coordinates of robots with series elastic actuators is a nontrivial task due to the increased dimensionality of the system and the increased degree of under-actuation. In one solution to this problem [11], Morris and Grizzle used the feedback linearization control technique [17] to exponentially stabilize the joint-angle control objectives.

The present paper proposes an alternative solution to the design of controllers which exponentially stabilize functions of the joint-angles in robots with series elastic actuators. Similar in structure to the "inner-outer loop" SEA control design methods [19], [18], the proposed method is a hierarchical control scheme; in the top tier of the control scheme, an idealized model of the robot is used to obtain "ideal joint torques" which exponentially stabilize control objectives computed on the joint-angle coordinates. A second tier controller is used to produce motor-rotor torques to achieve exponential convergence of the spring torques within the full SEA model to the corresponding ideal joint torques (communicated from the top tier).

The proposed method is distinguished from other similar control schemes through the establishment of conditions under which the motor-rotor torques produced by the hierarchical control scheme result in exponential stability of both the joint-angle outputs and the spring-deflection torques. Rapidly exponentially stabilizing control Lyapunov functions (RES-CLFs) [2], [3] are used to both construct the control laws in this paper and to establish stability of the hierarchical control scheme. Furthermore, it is shown that for a locally exponentially stable periodic orbit in the zero dynamics of the control system considered, the proposed hierarchical controller method renders a corresponding orbit in the full order SEA dynamics locally exponentially stable.

The paper is organized as follows. Section II gives the definition and an example construction of a rapidly exponentially stabilizing control Lyapunov function. Section III presents the proposed hierarchical SEA control strategy. Section IV establishes preliminary results on zero dynamics and orbits. Section V states the main result of the paper: exponential stability of the proposed control system. And Section VI gives results from simulation of the proposed controller.

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II. LYAPUNOV-BASED CONTROL

The problem of controlling a walking robot with series elastic actuators can be generalized to the following system in which the control objective is to drive $x \to 0$ (rapidly):

$$\dot{x} = f(x,z) + g(x,z)u, \quad (1)$$
$$\dot{z} = q(x,z)$$

where $x \in X$ are controlled states, $z \in Z$ are uncontrolled states and $u \in U$ are control inputs. Of prime interest are the *zero dynamics* $\dot{z} = q(0,z)$. Here it is assumed that the surface defined by the constraint $x \equiv 0$ is invariant. A general method for producing a u to achieve the control objective is through the construction of rapidly exponentially stabilizing control Lyapunov functions (RES-CLFs); the following definition of a RES-CLF can be found in [2], [3].

Definition 1: For the system (1)*, a continuously differentiable function $V_\varepsilon : X \to \mathbb{R}$ is a* **rapidly exponentially stabilizing control Lyapunov function (RES-CLF)** *if there exist positive constants $c_1, c_2, c_3 > 0$ such that for all* $0 < \varepsilon < 1$,

$$c_1\|x\|^2 \leq V_\varepsilon(x) \leq \frac{c_2}{\varepsilon^2}\|x\|^2, \quad (2)$$
$$\inf_{u\in U}\left[L_f V_\varepsilon(x,z) + L_g V_\varepsilon(x,z)u + \frac{c_3}{\varepsilon}V(x)\right] \leq 0, \quad (3)$$

for all $(x,z) \in X \times Z$.

The main result of this paper uses RES-CLFs to establish stability of control objectives for a class of robots with series elastic actuators.

A. Constructing Rapidly Exponentially Stabilizing Control Lyapunov Functions

In this paper, feedback linearization [17] control laws will be used on the full, nonlinear dynamics of the robot model to create linear input/output dynamical systems – once in this form, a RES-CLF for the outputs can be constructed by following the process described in [2], [3]. For an example RES-CLF derivation, consider the case when control system (1) takes the "normal form":

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}}_{F} x + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{G} u, \quad (4)$$

A RES-CLF for the outputs x can be constructed via:

$$V_\varepsilon(x) := x^T \underbrace{I_\varepsilon P I_\varepsilon}_{P_\varepsilon} x, \quad I_\varepsilon := \mathrm{diag}\left(\frac{1}{\varepsilon}I, I\right), \quad (5)$$

where I is the identity matrix and $P = P^T > 0$ solves the the continuous time algebraic Riccati equations (CARE) $F^T P + PF - PGG^T P + Q = 0$ for $Q = Q^T > 0$. The time-derivative of (5) is given by

$$\dot{V}_\varepsilon(x) = L_F V_\varepsilon(x) + L_G V_\varepsilon(x)u, \quad (6)$$

where

$$L_F V_\varepsilon(x) = x^T(F^T P_\varepsilon + P_\varepsilon F)x, \quad (7)$$
$$L_G V_\varepsilon(x) = 2x^T P_\varepsilon G, \quad (8)$$

are the Lie derivatives of $V_\varepsilon(x)$ along the vector fields F and G.

Lemma 1: (From [4]) For the dynamics (4) *and for P_ε defined in* (5)*, $V_\varepsilon(x) := x^T P_\varepsilon x$ is a RES-CLF with*

$$c_1 = \lambda_{\min}(P), \quad c_2 = \lambda_{\max}(P), \quad c_3 = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}.$$

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Proof: A proof of Lemma 1 is given in [4]. ■

B. Properties of Rapidly Exponentially Stabilizing Control Lyapunov Functions

Recall that the control objective is to stabilize the outputs to the origin, i.e. drive $x \to 0$ (rapidly). It has been shown [2], [3], that by using a RES-CLF, the outputs can be stabilized to the origin at a rate proportional to $\frac{1}{\varepsilon}$. In particular, consider the set of inputs u such that $\dot{V}_\varepsilon(x,u) \leq -\frac{c_3}{\varepsilon}V_\varepsilon(x)$; this set is called $K_\varepsilon(x)$ and is given by:

$$K_\varepsilon(x) := \{u : L_F V_\varepsilon(x) + L_G V_\varepsilon(x)u + \frac{c_3}{\varepsilon}V_\varepsilon(x) \leq 0\}.$$

Applying control inputs $u \in K_\varepsilon(x)$ to the output dynamics (4), with initial condition $x(0)$, results in:

$$\|x(t)\| \leq \frac{1}{\varepsilon}\sqrt{\frac{c_2}{c_1}}e^{-\frac{c_3}{2\varepsilon}t}\|x(0)\|. \quad (9)$$

That is, the norm of the control objectives converges to zero exponentially at a rate of $\frac{c_3}{2\varepsilon}$. Thus, rapidly exponentially stabilizing control Lyapunov functions can be used to determine inputs u to exponentially stabilize $V_\varepsilon(x)$ which implies exponential stability of the control objectives x.

C. Solutions and Periodic Orbits

The notation and definitions in [3] will be used to describe solutions and periodic orbits in the control system considered. In particular, let $\phi_t(x,z)$ be the solution of (1) with initial condition $(x,z) \in X \times Z$. The solution ϕ_t is called *periodic* with period $T > 0$ if $\phi_{(t+nT)}(x,z) = \phi_t(x,z)$ for all $n \in \mathbb{N}_{>0}$. A *periodic orbit* is denoted by the set $\mathcal{O} = \{\phi_t(x,z) \in X \times Z : 0 \leq t \leq T\}$ for a periodic solution ϕ_t. Define the distance from a point $p = (x,z) \in X \times Z$ to a periodic orbit $\mathcal{O}$ to be $\mathrm{dist}(p,\mathcal{O}) := \inf_{y\in\mathcal{O}}\|p - y\|$. Furthermore, denote a ball of radius $\delta > 0$ around a periodic orbit by the set $B_\delta(\mathcal{O}) = \{p = (x,z) \in X \times Z : \mathrm{dist}(p,\mathcal{O}) < \delta\}$. A periodic orbit $\mathcal{O}$ is *locally exponentially stable* if there exist $\delta, M, \beta > 0$ such that if $(x,z) \in B_\delta(\mathcal{O})$ it follows that $\mathrm{dist}(\phi_t(x,z),\mathcal{O}) \leq Me^{-\beta t}\mathrm{dist}((x,z),\mathcal{O})$.

Similarly, we denote the flow of the zero dynamics $\dot{z} = q(0,z)$ by ϕ_t^z and for a periodic flow we denote the corresponding periodic orbit $\mathcal{O}_Z \subset Z$. Due to the assumption that the zero dynamics surface Z is invariant, a periodic orbit for the zero dynamics, $\mathcal{O}_Z$, corresponds to a periodic orbit for the full-order dynamics, $\mathcal{O} = \iota_0(\mathcal{O}_Z)$, through the canonical embedding $\iota_0 : Z \to X \times Z$ given by $\iota_0(z) = (0,z)$.

III. Series Elastic Actuator Control

Consider the model of a robot with N rigid links arranged in a tree structure and connected by $N-1$ actuated revolute joints. Let $\theta_a \in Q_a \subset \mathbb{R}^{N-1}$ be a vector of "actuated joint angles" which describe the relative angles between consecutive rigid links in the robot. Assume that the free end of one of the links in robot is connected to a fixed world frame, e.g. the ground, through a passive revolute joint, forming an angle $\theta_b \in \mathbb{R}$. Combine the passive and actuated joint angles to form a vector of body, or shape, coordinates[1] $\theta = (\theta_b, \theta_a) \in Q \subset \mathbb{R}^N$ which are sometimes referred to as "joint angles".

In the case when the robot is driven by series-elastic actuators, let $\theta_m \in Q_m \subset \mathbb{R}^{N-1}$ be a vector of "motor angles" describing the orientation of each motor rotor. Given this choice of coordinates, the dynamics model of a robot with series-elastic actuators can be expressed as:

$$D(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = BK(\theta_m - \theta_a), \tag{10}$$

$$J\ddot{\theta}_m + K(\theta_m - \theta_a) = \tau, \tag{11}$$

where the left side of (10) are the standard robot manipulator dynamics (see [12]), with inertia matrix $D(\theta)$, Coriolis matrix $C(\theta,\dot{\theta})$ and gravity vector $G(\theta)$. The bottom equation (11) describes the motor dynamics with diagonal matrices of motor rotor inertia J and spring stiffnesses, K, and motor torques $\tau \in \mathbb{R}^{N-1}$. Note that the manipulator and the motor dynamics are coupled through the spring-deflection torques, $K(\theta_m - \theta_a)$, which act on the manipulator dynamics through B, a matrix consisting of a row of $1 \times N-1$ zeros stacked on an $N-1 \times N-1$ identity matrix.

This section presents a solution to the control problem of stabilizing functions (called outputs) of the joint angles, $y(\theta) \to 0$, through the specification of motor-rotor torques. The control strategy is hierarchical: an idealized model of the robot is used to construct a RES-CLF to achieve convergence of the joint-angle outputs using ideal joint torques. To overcome the discrepancy between this ideal model and the full SEA dynamics – namely, a mismatch between ideal and spring deflection torques – a second RES-CLF will be created on the SEA model to obtain motor-rotor inputs which drive the spring torques to their corresponding ideal joint torque values. A linear combination of these two RES-CLFs will be used to establish stability for both the joint-angle outputs and the spring-torque outputs simultaneously.

A. Joint-Angle Control (via Ideal Actuators)

In the first tier of the hierarchical control approach, an ideal-actuator model is obtained by replacing the spring torques $K(\theta_m - \theta_a)$ in (10) with a vector of ideal torque sources $u_r \in \mathbb{R}^{N-1}$:

$$D(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = Bu_r. \tag{12}$$

[1]Note that in general it may be necessary to consider a reduced subset $\tilde{Q}$ of Q to ensure that the employed functions of θ are well defined.

These dynamics can also be expressed as an affine control system of the form

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = f_r(\theta,\dot{\theta}) + g_r(\theta)u_r, \tag{13}$$

with

$$f_r(\theta,\dot{\theta}) = \begin{bmatrix} \dot{\theta} \\ D^{-1}(\theta)(-C(\theta,\dot{\theta})\dot{\theta} - G(\theta)) \end{bmatrix} \tag{14}$$

$$g_r(\theta) = \begin{bmatrix} 0 \\ D^{-1}(\theta)B \end{bmatrix}. \tag{15}$$

Let $y : Q \to \mathbb{R}^{N-1}$ be a vector of *joint angle outputs* which encode the control objective $y(\theta) \to 0$. In the case of ideal actuation (12), the joint-angle outputs have vector relative degree two [17] and the relationship between input u_r and output $y(\theta)$ can be written

$$\ddot{y} = \underbrace{L_{f_r}L_{f_r}y(\theta,\dot{\theta})}_{L^2_{f_r}(\theta,\dot{\theta})} + \underbrace{L_{g_r}L_{f_r}y(\theta)}_{A(\theta)}u_r. \tag{16}$$

Here L_{f_r} and L_{g_r} are Lie derivatives [17] and the decoupling matrix, $A(\theta)$, is assumed to be invertible through proper choice of $y(\theta)$. Applying the feedback linearization [17] control law:

$$u_r = A^{-1}(\theta)(-L^2_{f_r}(\theta,\dot{\theta}) + \mu_r), \tag{17}$$

to (16) yields $\ddot{y} = \mu_r$. Defining $x_r := (y(\theta), \dot{y}(\theta,\dot{\theta})) \in X_r$, it follows that

$$\dot{x}_r = \underbrace{\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}}_{F_r} x_r + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{G_r} \mu_r. \tag{18}$$

Noting that (18) is similar in form to (4), a RES-CLF for the joint-angle outputs can be constructed using the method described in Section II-A. In particular,

$$V_{\varepsilon_r}(x_r) := x_r^T P_{\varepsilon_r} x_r, \tag{19}$$

where P_{ε_r} is obtained through the CARE on the control system (F_r, G_r) and with $Q = I$.

For $\mu_r^* \in K^2_{\varepsilon_r}$, which is defined to be the set of twice differentiable controllers such that $\mu_r^*(x_r) \in K_{\varepsilon_r}(x_r)$, denote the corresponding ideal-actuator torques u_r^* and time-derivative of (19), $\dot{V}^*_{\varepsilon_r}(x_r, \mu_r^*)$,

$$u_r^* := A^{-1}(\theta)(-L^2_{f_r}(\theta,\dot{\theta}) + \mu_r^*), \tag{20}$$

$$\dot{V}^*_{\varepsilon_r}(x_r,\mu_r^*) := L_F V_{\varepsilon_r}(x_r) + L_G V_{\varepsilon_r}(x_r)\mu_r^*. \tag{21}$$

These ideal-actuator torques u_r^* will be used as the control target of the isolated motor control system. As the particular choice of motor control law uses both $\dot{u}_r^*$ and $\ddot{u}_r^*$, we require that u_r^* be twice differentiable and hence, μ_r^* must also be twice differentiable. One choice of $\mu_r^* \in K^2_{\varepsilon_r}$ for the systems in this paper is $\mu_r^* = -2\varepsilon_r\dot{y} - \varepsilon_r^2 y$, which is commonly used in feedback linearization methods [17] to place the poles of the closed loop system, obtained after applying μ_r^* to (18), at $-\varepsilon_r$.

B. Joint-Angle Dynamics with Series Elastic Actuators

Now consider the joint-angle output dynamics under the series elastic actuator dynamics model (10). When the spring torques $K(\theta_m - \theta_a)$ do not equal the ideal actuator torques u_r^*, the joint-angle outputs will not evolve in the same manner as they did under ideal actuation. In particular, (16) becomes

$$\ddot{y} = L_{f_r}^2(\theta, \dot{\theta}) + A(\theta)K(\theta_m - \theta_a), \tag{22}$$

and thus the dynamics on x_r (using mixed notation at this point in the paper) become:

$$\dot{x}_r = F_r x_r + G_r \mu_r^* + G_r A(\theta)(K(\theta_m - \theta_a) - u_r^*). \tag{23}$$

Furthermore, $V_{\varepsilon_r}(x_r)$ under series elastic actuation can no longer be made to be a RES-CLF, as its time derivative is now:

$$\dot{V}_{\varepsilon_r} = \dot{V}_{\varepsilon_r}^*(x_r, \mu_r^*) + L_G V_{\varepsilon_r}(x_r) A(\theta)(K(\theta_m - \theta_a) - u_r^*) \tag{24}$$

These alterations to the joint-angle output dynamics associated with series elastic actuation motivate the construction of an additional control law to drive $K(\theta_m - \theta_a) \to u_r^*$.

C. Spring-Deflection (SEA-Motor) Control

The goal of this section is to create a motor-rotor torque control law which drives the spring deflection torques to corresponding ideal actuator torque values. To this end, define $y_m : Q_m \times Q_a \times \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$, a vector of *motor control objectives*, as follows:

$$y_m(\theta_m, \theta_a, u_r^*) = K(\theta_m - \theta_a) - u_r^*. \tag{25}$$

For the series-elastic actuator motor dynamics, given in (11), picking the motor control law

$$\tau = J(K^{-1}(\mu_m + \ddot{u}_r^*) + \ddot{\theta}_a) + K(\theta_m - \theta_a), \tag{26}$$

results in $\ddot{y}_m = \mu_m$. As with the robot output states, define the following motor output states $x_m := (y_m, \dot{y}_m) \in X_m$. Applying (26) to (11) results in the following dynamics of the motor outputs:

$$\dot{x}_m = \underbrace{\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}}_{F_m} x_m + \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{G_m} \mu_m. \tag{27}$$

Using the method described in Section II-A, a RES-CLF for the motor outputs x_m can be constructed via:

$$V_m(x_m) := x_m^T P_{\varepsilon_m} x_m, \tag{28}$$

where P_{ε_m} is obtained through the CARE on the control system (F_m, G_m) and with $Q = I$. Thus, applying the motor control law (26), with $\mu_m \in K_{\varepsilon_m}(x_m)$ and $\ddot{u}_r^*$ obtained by differentiating (20) twice, to the motor dynamics model (11), results in exponential convergence of the SEA spring torques to the ideal joint torques (20). The next section presents conditions under which this control law also results in simultaneous convergence of the joint-angle outputs.

IV. Analysis of Rigid and SEA Systems

Towards the goal of stating the main result of the paper, we begin by first establishing that the zero dynamics of the series-elastic actuator control system are identical to the zero dynamics within the ideal-actuator control system. With this result in place, we then establish that locally exponentially stable periodic orbits in the zero dynamics correspond to invariant sets of the solutions to the series-elastic actuator control system. These preliminary results are used in the proof of the main result.

A. Preliminary Result: Equivalence of Zero Dynamics

In Section III-A, it was shown that applying the feedback control law (17) to the ideal actuator dynamics given in (12) results in a linear control system (18) on the joint-angle outputs, x_r. As the dimension of the vector x_r is $2N-2$ and based on the vector relative degree assumption, by the Frobenius theorem [17], there exists a 2 dimensional vector $z_r = (\eta_1(\theta, \dot{\theta}), \eta_2(\theta, \dot{\theta}))$ such that the change of coordinates $\Phi_r : (\theta, \dot{\theta}) \mapsto (x_r, z_r)$ is a diffeomorphism. By construction, in the ideal actuator system (13), η_1 and η_2 satisfy

$$d\eta_i(\theta, \dot{\theta}) g_r(\theta) \equiv 0, \quad \forall i \in \{1, 2\}. \tag{29}$$

Using the explicit form of $g_r(\theta)$, given in (15), combining the two equations in (29), and noting that $dz_r = (d\eta_1, d\eta_2)$, it follows that

$$L_{g_r} z_r(\theta, \dot{\theta}) = \frac{\partial z_r(\theta, \dot{\theta})}{\partial \dot{\theta}} D^{-1}(\theta) B \equiv 0. \tag{30}$$

Thus, for the ideal actuator system, the time derivative of z_r is given by

$$\begin{aligned} \dot{z}_r &= L_{f_r} z_r(\theta, \dot{\theta}) \\ &= \frac{\partial z_r(\theta, \dot{\theta})}{\partial \theta} \dot{\theta} - \frac{\partial z_r(\theta, \dot{\theta})}{\partial \dot{\theta}} D^{-1}(\theta)(C(\theta, \dot{\theta})\dot{\theta} + G(\theta)). \end{aligned} \tag{31}$$

Using the diffeomorphism $(\theta, \dot{\theta}) = \Phi_r^{-1}(x_r, z_r)$, define the function $q_r(x_r, z_r) := L_{f_r} z_r(\Phi_r^{-1}(x_r, z_r))$, so that in the new coordinates (x_r, z_r) the ideal actuator system equations (13) read

$$\begin{aligned} \dot{x}_r &= F_r x_r + G_r \mu_r, \\ \dot{z}_r &= q_r(x_r, z_r). \end{aligned} \tag{32}$$

The zero dynamics, $\dot{z}_r = q_r(0, z_r)$, of the ideal-actuator control system are obtained through the constraint $x_r \equiv 0$. The goal of the next two Propositions is to establish a relationship between the zero dynamics in the ideal system and the zero dynamics of the series elastic actuator system.

Proposition 1: For the series-elastic actuator system given in (10) *and* (11)*, the function*

$$\Phi : (\theta, \dot{\theta}, \theta_m, \dot{\theta}_m) \mapsto (x_r, x_m, z_r) \tag{33}$$

is a C^∞-local diffeomorphism.

Proof: Following [20], from the Inverse Function Theorem, Φ is a C^∞-local diffeomorphism if and only if the matrix

$$d\Phi = \begin{bmatrix} dy \\ d\dot{y} \\ dy_m \\ d\dot{y}_m \\ dz_r \end{bmatrix}, \tag{34}$$

has full rank.

From the vector relative degree assumption and by the fact that $\Phi_r : (\theta, \dot{\theta}) \mapsto (x_r, z_r)$ is a diffeomorphism, it follows that $dy, d\dot{y}, dz_r$ are linearly independent. Furthermore, as y_m and $\dot{y}_m$ are functions of the motor coordinates θ_m and $\dot{\theta}_m$, while y, $\dot{y}$, and z_r are not, it follows that $dy, d\dot{y}, dy_m, d\dot{y}_m, dz_r$ are linearly independent. Thus, $d\Phi$ has full rank and by the Inverse Function Theorem, Φ is a C^∞-local diffeomorphism. ■

Using the diffeomorphism established in Proposition 1, in the x_r, x_m, z_r coordinates the series-elastic actuator system equations (10) and (11) become

$$\dot{x}_r = F_r x_r + G_r \mu_r^* + G_r A(x_r, z_r) x_m, \tag{35}$$

$$\dot{x}_m = F_m x_m + G_m \mu_m, \tag{36}$$

$$\dot{z}_r = q(x_r, x_m, z_r). \tag{37}$$

where (35) is a restatement of (23) now with consistent notation. Note that the diffeomorphism $(\theta, \dot{\theta}) = \Phi_r^{-1}(x_r, z_r)$ is used to express the decoupling matrix as a function of x_r and z_r, i.e. $A(x_r, z_r)$. Furthermore, the z_r dynamics are (currently) assumed to be a function of x_r, x_m, and z_r.

Proposition 2: The zero dynamics for the series-elastic actuator system are equivalent to the zero dynamics for the ideal-actuator system, i.e. $q(0, 0, z_r) = q_r(0, z_r)$.

Proof: We will prove the more general result that $q(x_r, x_m, z_r) = q_r(x_r, z_r)$ by computing $q(x_r, x_m, z_r) := \dot{z}_r$ and showing that the result is equivalent to (31). Note that

$$\dot{z}_r = \frac{\partial z_r(\theta, \dot{\theta})}{\partial \theta} \dot{\theta} + \frac{\partial z_r(\theta, \dot{\theta})}{\partial \dot{\theta}} \ddot{\theta}. \tag{38}$$

In series-elastic actuator model,

$$\ddot{\theta} = D^{-1}(\theta)(BK(\theta_m - \theta_a) - C(\theta, \dot{\theta})\dot{\theta} - G(\theta)), \tag{39}$$

and thus

$$\begin{aligned} \dot{z}_r = & \frac{\partial z_r(\theta, \dot{\theta})}{\partial \theta} \dot{\theta} - \frac{\partial z_r(\theta, \dot{\theta})}{\partial \dot{\theta}} D^{-1}(\theta)(C(\theta, \dot{\theta})\dot{\theta} + G(\theta)) \\ & + \frac{\partial z_r(\theta, \dot{\theta})}{\partial \dot{\theta}} D^{-1}(\theta) BK(\theta_m - \theta_a). \end{aligned} \tag{40}$$

Recalling from (30) by construction $\frac{\partial z_r(\theta, \dot{\theta})}{\partial \dot{\theta}} D^{-1}(\theta) B \equiv 0$, the bottom term in (40) vanishes, leaving

$$\dot{z}_r = \frac{\partial z_r(\theta, \dot{\theta})}{\partial \theta} \dot{\theta} - \frac{\partial z_r(\theta, \dot{\theta})}{\partial \dot{\theta}} D^{-1}(\theta)(C(\theta, \dot{\theta})\dot{\theta} + G(\theta))$$

which is exactly (31). Hence $q(x_r, x_m, z_r) = q_r(x_r, z_r)$ and thus the zero dynamics $\dot{z}_r = q(0, 0, z_r)$ and $\dot{z}_r = q_r(0, z_r)$ are equivalent. ■

B. Preliminary Result: Periodic Orbits

To establish the main result, we must first show that solutions of the series-elastic actuator control system (35)–(37) are bounded on the domain of interest. Here, we establish the domain of interest as an invariant set around a periodic orbit in the full SEA system. Proposition 2 and Theorem 1 of [3] are used to produce a periodic orbit in the full SEA system. Theorem 2 of [4] is used to construct an invariant set containing the periodic orbit.

For the ideal-actuator control system, (32), let $\mathcal{O}_Z$ be a locally exponentially stable periodic orbit for the zero dynamics $\dot{z}_r = q_r(0, z)$. As shown in the Section II-C, $\mathcal{O}_Z$ corresponds to a periodic orbit, $\mathcal{O}_r = \iota_0^r(\mathcal{O}_Z)$, for the ideal-actuator dynamics through the canonical embedding $\iota_0^r : Z_r \to X_r \times Z_r$ given by $\iota_0^r(z) = (0, z)$. By Theorem 1 of [3], $\mathcal{O}_r = \iota_0^r(\mathcal{O}_Z)$ is a locally exponentially stable periodic orbit of the closed-loop system (32) with $\mu_r \in K_{\varepsilon_r}^2$. Furthermore, as Proposition 2 established equivalence of the zero dynamics in the ideal and the SEA systems, a periodic orbit for the zero dynamics, $\mathcal{O}_Z$, also corresponds to a periodic orbit for the series-elastic actuator dynamics, $\mathcal{O}_S = \iota_0^s(\mathcal{O}_Z)$, through the canonical embedding $\iota_0^s : Z_r \to X_r \times X_m \times Z_r$ given by $\iota_0^s(z_r) = (0, 0, z_r)$.

The current goal is to establish an invariant set around the orbit $\mathcal{O}_S = \iota_0^s(\mathcal{O}_Z)$ in the SEA control system. To this end, the distance of the solution of the SEA control system to the set $\mathcal{O}_S$ will be related to the distance of the solution to the ideal system to the corresponding locally exponentially stable orbit $\mathcal{O}_r$ through Tichinov's theorem on singularly perturbed systems [9].

Lemma 2: Let $\mathcal{O}_Z$ be a locally exponentially stable periodic orbit for the zero dynamics $\dot{z}_r = q_r(0, z)$, and let $\mathcal{O}_r = \iota_0^r(\mathcal{O}_Z)$ and $\mathcal{O}_S = \iota_0^s(\mathcal{O}_Z)$ be the corresponding periodic orbits for the ideal actuator and the series-elastic actuator systems, respectively.

Consider the closed-loop behavior obtained after applying the motor-torque control law (26)*, with Lipschitz continuous $\mu_m(x_m) \in K_{\varepsilon_m}(x_m)$ and the ideal-actuator control law* (20)*, to the series-elastic actuator system given in* (10) *and* (11) *with an initial condition (x_r^*, x_m^*, z_r^*) in a neighborhood of $\mathcal{O}_S$, described by the flow $\phi_t^s = \phi_t^s(x_r^*, x_m^*, z_r^*)$. It follows that there exists an $\varepsilon_r^* > 0$ such that*

$$\mathrm{dist}(\phi_t^s, \mathcal{O}_S) \leq M\delta + O(\varepsilon_r). \tag{41}$$

for all $0 \leq \varepsilon_r \leq \varepsilon_r^*$ and all $t \geq 0$, and where M and δ are the stability parameters of $\mathcal{O}_r$. ⌟

Proof: Let $\phi_t^r = \phi_t^r(x_r^*, z_r^*)$ be the solution to the ideal-actuator control system equations (32) with the initial condition $\phi_0^r = (x_r^*, z_r^*)$. It follows from Tichinov's theorem on singularly perturbed systems [9] and the RES-CLFs constructed in Section III that the norm of solutions in the SEA control system is bounded above by the norm of solutions in the ideal-actuator system. In particular,

$$\|\phi_t^s\| \leq \|\phi_t^r\| + O(\varepsilon_r) \tag{42}$$

Because $\mathscr{O}_r$ is a locally exponentially stable periodic orbit,

$$\text{dist}(\phi_t^r, \mathscr{O}_r) \leq Me^{-\alpha t}\text{dist}(\phi_0^r, \mathscr{O}_r), \tag{43}$$

with M and α stability parameters of $\mathscr{O}_r$. Letting the stability parameter δ of $\mathscr{O}_r$ be $\delta = \text{dist}(\phi_0^r, \mathscr{O}_r)$ and noting that the maximum of $Me^{-\alpha t}\text{dist}(\phi_0^r, \mathscr{O}_r)$ occurs at $t = 0$, (43) becomes

$$\text{dist}(\phi_t^r, \mathscr{O}_r) \leq M\text{dist}(\phi_0^r, \mathscr{O}_r) = M\delta. \tag{44}$$

Also from Tichinov's theorem,

$$\text{dist}(\phi_t^s, \mathscr{O}_S) \leq \text{dist}(\phi_t^r, \mathscr{O}_r) + O(\varepsilon_r). \tag{45}$$

Thus, combining (44) and (45),

$$\text{dist}(\phi_t^s, \mathscr{O}_S) \leq M\delta + O(\varepsilon_r), \tag{46}$$

Lemma 2 has been verified. ■

Let $\phi_t^s = \phi_t^s(x_r^*, x_m^*, z_r^*)$ be the solution of the SEA control system dynamics under the assumptions in Lemma 2. A consequence of Lemma 2 is that the set

$$\Omega = \{\phi_t^s : \text{dist}(\phi_t^s, \mathscr{O}_S) \leq M\delta + O(\varepsilon_r) \tag{47}$$

is invariant in the series-elastic actuator control system. This property will be leveraged in the proof of the main result of the paper.

V. MAIN-RESULT

This section presents two theorems describing the main result of the paper. In Theorem 1, it is shown that for proper choice of control parameters, the proposed hierarchical control approach results in exponential stability of the entire series elastic actuator system through simultaneous convergence of the joint-angle and the motor control objectives. Theorem 2 is a natural consequence of the result from Theorem 1 together with Theorem 1 of [3]. In particular, Theorem 2 establishes local exponential stability of the orbit in the full series-elastic actuator system, $\mathscr{O}_S = \iota_0^s(\mathscr{O}_Z)$, obtained from a locally exponentially stable orbit $\mathscr{O}_Z$ for the zero dynamics $\dot{z}_r = q_r(0, z)$.

To establish simultaneous convergence of the joint-angle outputs and the motor outputs, define $V_c : X_r \times X_m \to \mathbb{R}$ as follows:

$$V_c(x_r, x_m) := V_{\varepsilon_r}(x_r) + \rho V_m(x_m), \tag{48}$$

where $\rho > 0$ is to be specified. It will be shown that under the assumptions made in Theorem 1, $V_c(x_r, x_m)$ can be made exponentially stable for *any* twice differentiable "high-level" control law of the form (20), stated again for reference:

$$u_r^* := A^{-1}(\theta)(-L_{f_r}^2(\theta, \dot{\theta}) + \mu_r^*), \tag{49}$$

with $\mu_r^* \in K_{\varepsilon_r}^2$.

Theorem 1: Let $\mathscr{O}_Z$ be a locally exponentially stable periodic orbit for the zero dynamics $\dot{z}_r = q_r(0, z)$, and let $\mathscr{O}_S = \iota_0^s(\mathscr{O}_Z)$ be the corresponding periodic orbit for the series-elastic actuator system.

Consider the closed-loop behavior obtained after applying the motor-torque control law (26)*, with Lipschitz continuous $\mu_m(x_m) \in K_{\varepsilon_m}(x_m)$ and the ideal-actuator control law* (49) *with $\mu_r^* \in K_{\varepsilon_r}^2$, to the series-elastic actuator system given in* (10) *and* (11)*. There exist positive constants $c_{1c}, c_{2c}, c_{3c} > 0$, an $\bar{\varepsilon}_m > 0$ and a $\rho > 0$ such that the composite Lyapunov function* (48) *satisfies*

$$c_{1c}(\|x_r\|^2 + \|x_m\|^2) \leq V_c(x_r, x_m) \leq \frac{c_{2c}}{\varepsilon_r^2}(\|x_r\|^2 + \|x_m\|^2),$$
$$\dot{V}_c(x_r, x_m, \mu_m) \leq -\frac{c_{3c}}{\varepsilon_r}V_c(x_r, x_m),$$

for all $0 < \varepsilon_m \leq \bar{\varepsilon}_m$ and all $0 < \varepsilon_r \leq 1$.

Proof: To establish the first set of inequalities, note that by definition (48), $V_c(x_r, x_m)$ is a linear combination of two RES-CLFs, (19) and (28) – both of which satisfy (2) by definition. Thus, straightforward computation yields

$$c_{1r}\|x_r\|^2 + \rho c_{1m}\|x_m\|^2 \leq V_c(x_r, x_m), \tag{50}$$
$$V_c(x_r, x_m) \leq \frac{c_{c_{2r}}}{\varepsilon_r^2}\|x_r\|^2 + \frac{\rho c_{c_{2m}}}{\varepsilon_m^2}\|x_m\|^2. \tag{51}$$

For ε_m and ρ satisfying:

$$\frac{\rho}{\varepsilon_m^2} \leq \frac{c_{2r}}{c_{2m}\varepsilon_r^2}, \tag{52}$$

it follows that

$$V_c(x_r, x_m) \leq \frac{c_{c_{2r}}}{\varepsilon_r^2}(\|x_r\|^2 + \|x_m\|^2). \tag{53}$$

Picking ε_m and ρ satisfying (52), and $c_{1c} = \min(c_{1r}, \rho c_{1m})$ and $c_{2c} = c_{2r}$, the inequalities given in (50) and (53) establish the first condition of the Theorem.

The proof of the inequality on $\dot{V}_c$ uses the property (3) of the RES-CLFs (19) and (28), together with Lemma 1. From the assumptions in the Theorem, $\mu_m(x_m) \in K_{\varepsilon_m}(x_m)$ and $\mu_r^* \in K_{\varepsilon_r}^2$; and therefore, after control is applied,

$$\dot{V}_{\varepsilon_r}^*(x_r, \mu_r^*) \leq -\frac{c_{3r}}{\varepsilon_r}V_{\varepsilon_r}(x_r) \leq -\frac{c_{1r}c_{3r}}{\varepsilon_r}\|x_r\|^2. \tag{54}$$
$$\dot{V}_m(x_m, \mu_m) \leq -\frac{c_{3m}}{\varepsilon_m}V_m(x_m) \leq -\frac{c_{1m}c_{3m}}{\varepsilon_m}\|x_m\|^2. \tag{55}$$

Thus, $\dot{V}_c(x_r, x_m, \mu_r^*, \mu_m)$ satisfies

$$\dot{V}_c(x_r, x_m, \mu_r^*, \mu_m) \leq -\frac{c_{1r}c_{3r}}{\varepsilon_r}\|x_r\|^2 - \rho\frac{c_{1m}c_{3m}}{\varepsilon_m}\|x_m\|^2 + 2x_r^T P_{\varepsilon_r} GA(x_r, z_r)Tx_m. \tag{56}$$

By the Cauchy-Schwarz inequality,

$$x_r^T P_{\varepsilon_r} GA(x_r, z_r)Tx_m \leq \|x_r\|\,\|P_{\varepsilon_r}\|\,\|G\|\,\|A(x_r, z_r)\|\,\|T\|\,\|x_m\|. \tag{57}$$

From the definition of P_{ε_r} in (19) we have

$$\|P_{\varepsilon_r}\| \leq \frac{1}{\varepsilon_r^2}\|P_r\| \leq \frac{1}{\varepsilon_r^2}\lambda_{\max}(P_r) = \frac{c_{2r}}{\varepsilon_r^2}, \tag{58}$$

and noting that $\|G\| = \|T\| = 1$,

$$x_r^T P_{\varepsilon_r} G A(x_r) T x_m \leq \frac{c_{2r}}{\varepsilon_r^2} \|x_r\| \, \|A(x_r, z_r)\| \, \|x_m\|.$$

To establish a bound on $\|A(x_r, z_r)\|$, we use the fact that the norm of a matrix A is less than or equal its maximum singular value, denoted $\sigma_{\max}(A)$ [7]. Therefore, using the closed set Ω established through Lemma 2, let

$$\sigma^* = \max_{(x_r, x_m, z_r) \in \Omega} \sigma_{\max}(A(x_r, z_r)), \tag{59}$$

it follows that

$$\|A(x_r, z_r)\| \leq \sigma_{\max}(A(x_r, z_r)) \leq \sigma^*, \quad (x_r, x_m, z_r) \in \Omega.$$

Thus

$$x_r^T P_{\varepsilon_r} G A(x_r) T x_m \leq \frac{c_{2r}\sigma^*}{\varepsilon_r^2} \|x_r\| \|x_m\|.$$

Now (56) becomes

$$\dot{V}_c(x_r, x_m, \mu_r^*, \mu_m) \leq -\frac{c_{1r}c_{3r}}{\varepsilon_r}\|x_r\|^2 - \rho \frac{c_{1m}c_{3m}}{\varepsilon_m}\|x_m\|^2 + \frac{2c_{2r}\sigma^*}{\varepsilon_r^2}\|x_r\|\|x_m\|.$$

Following [9], [3], define $\psi(x_r, x_m) := (\|x_r\|, \|x_m\|)$ and

$$\Lambda := \begin{bmatrix} c_{1r}c_{3r} & -\frac{c_{2r}\sigma^*}{\varepsilon_r} \\ -\frac{c_{2r}\sigma^*}{\varepsilon_r} & \frac{\rho c_{1m} c_{3m} \varepsilon_r}{\varepsilon_m} \end{bmatrix}, \tag{60}$$

so that

$$\dot{V}_c(x_r, x_m, \mu_r^*, \mu_m) \leq -\frac{1}{\varepsilon_r}\psi^T(x_r, x_m)\Lambda\psi(x_r, x_m). \tag{61}$$

For exponential convergence, we need $|\Lambda| > 0$; this is equivalent to

$$\frac{\rho c_{1r}c_{3r}c_{1m}c_{3m}\varepsilon_r}{\varepsilon_m} - \left(\frac{c_{2r}\sigma^*}{\varepsilon_r}\right)^2 > 0. \tag{62}$$

Rearranging (62) we obtain an expression in terms of the variables of interest in this Theorem:

$$\frac{\rho}{\varepsilon_m} > \frac{(c_{2r}\sigma^*)^2}{c_{1r}c_{3r}c_{1m}c_{3m}} \frac{1}{\varepsilon_r^3}. \tag{63}$$

Thus, choosing

$$\bar{\varepsilon}_m < \rho \varepsilon_r^3 \frac{c_{1r}c_{3r}c_{1m}c_{3m}}{(c_{2r}\sigma^*)^2}, \tag{64}$$

it follows that for any $0 < \varepsilon_m \leq \bar{\varepsilon}_m$, $|\Lambda| > 0$ and thus $V_c(x_r, x_m)$ is exponentially stabilized at a rate proportional to $\frac{1}{\varepsilon_r}$. ■

The following Theorem states the main result from the paper, which is a natural corollary of the application of Theorem 1 of [3] together with establishment of a Lyapunov function for the full series-elastic actuator dynamics, as shown in Theorem 1 of the current paper.

Theorem 2: Let $\mathcal{O}_Z$ *be a locally exponentially stable periodic orbit for the zero dynamics* $\dot{z}_r = q_r(0, z)$*, and let* $\mathcal{O}_S = \iota_0^s(\mathcal{O}_Z)$ *be the corresponding periodic orbit for the series-elastic actuator system. Applying the motor-torque control law* (26)*, with Lipschitz continuous* $\mu_m(x_m) \in K_{\varepsilon_m}(x_m)$ *and the ideal-actuator control law* (49) *with* $\mu_r^* \in K_{\varepsilon_r}^2$*, to the series-elastic actuator system given in* (10) *and* (11)*, renders the periodic orbit for the series-elastic actuator system,* $\mathcal{O}_S$*, locally exponentially stable.*

VI. Simulation Results

To demonstrate the controller developed in this paper, we implement it on the pendulum on a cart system (see Fig. 1). In particular, the configuration of the system consists of two variables, $\theta \in (-\pi, \pi)$ and $x \in \mathbb{R}$. In this case, in order to create a stable periodic orbit in the zero dynamics, we assume that the cart is forced by a forcing function:

$$F(x, \dot{x}) = (m + M)(-x + \dot{x}(1 - x^2 - \dot{x}^2)), \tag{65}$$

where M is the mass of the cart and m is the mass of the pendulum. In the case of ideal actuation, this results in dynamics of the form given in (12) as shown in Table I; here l is the length of the pendulum and g is the acceleration due to gravity. In addition, note that we will assume that we only have control authority over the pendulum, and thus $B = (0, 1)^T$. The control objective is to stabilize the pendulum to the upright position; therefore, we pick $y = \theta$.

For this system, we construct the zero dynamics coordinates according to the methods outlined in [20]. In particular, this results in zero dynamic coordinates given by $z = (\eta_1, \eta_2)$ with

$$\begin{aligned} \eta_1 &= x, \\ \eta_2 &= (m + M)\dot{x} - lm\cos(\theta)\dot{\theta}. \end{aligned}$$

It is easy to verify that this choice of zero dynamics coordinates results in satisfaction of (30). The end result is, therefore, zero dynamics of the form given in (1). By construction of the forcing function (65), it follows that $\dot{z} = q(0, z)$ has a locally exponentially stable periodic orbit.

For this system, we will consider two cases: the ideal actuator case and the SEA case. In both cases, we show that the controllers presented in this paper, in stabilizing $y \to 0$, result in locally exponentially stable periodic orbits.

Ideal actuator case: Simulation results for the ideal actuator case, i.e., the case where the dynamics are governed by (12) are shown in Fig. 2 in red. In this case, as expected, it can be seen that the stable periodic orbit for the zero dynamics

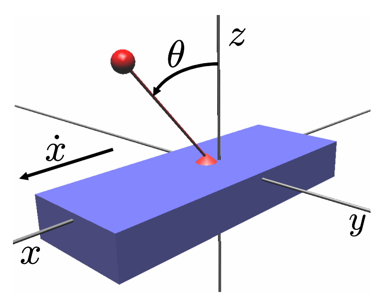

Fig. 1. The pendulum on a cart system used to demonstrate the formal results of the paper. In this case, the pendulum is actuated and the system is considered both in the case of ideal actuation and series elastic actuation.

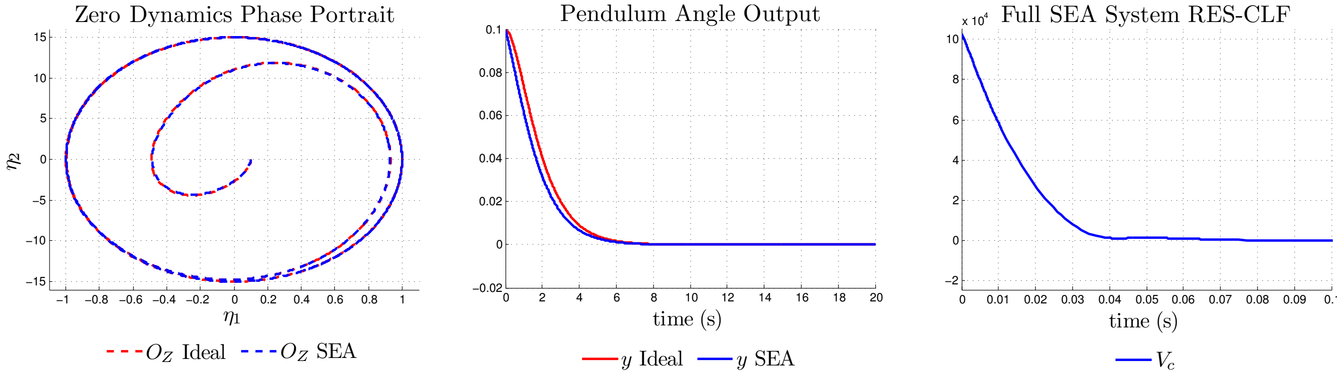

Fig. 2. Simulation results including (left) periodic orbit that is locally exponentially stable, (middle) convergence of the outputs for both the idea and SEA system, (right) convergence of the composite Lyapunov function (48) in the case of SEA.

$$D(x,\theta) = \begin{bmatrix} m+M & -lm\cos(\theta) \\ -lm\cos(\theta) & l^2 m \end{bmatrix}, \qquad C(\dot{x},\dot{\theta}) \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + G(\theta) = \begin{bmatrix} (m+M)x - (m+M)\dot{x} + (m+M)x^2\dot{x} + (m+M)\dot{x}^3 + lm\sin(\theta)\dot{\theta}^2 \\ -glm\sin(\theta) \end{bmatrix}$$

TABLE I

IDEAL DYNAMICS OF THE PENDULUM ON A CART WITH THE FORCING FUNCTION F (65).

implies the existence of a stable periodic orbit for the full-order dynamics (left plot in Fig. 2). In addition, the outputs y converge exponentially to zero.

SEA case: To demonstrate the main results of this paper, specifically Theorem 1 and Theorem 2, it is assumed that the pendulum is actuated through as SEA as governed by (10) and (11). In this case, controllers satisfying the conditions of Theorem 1 are chosen to stabilize the output y. The end result, as predicted by Theorem 2, is a stable periodic orbit for the full-order SEA dynamics (as shown in Fig. 2). The convergence of the output y in this case is also shown in 2. Finally, the behavior of the composite Lyapunov function (48) is shown in the right plot of 2. As predicted by Theorem 1, it converges to zero exponentially for the full-order system.

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