Distributed Quadratic Programming-Based Nonlinear Controllers for Periodic Gaits on Legged Robots

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***Abstract*—Quadratic programming (QP)-based nonlinear controllers have gained increasing popularity in the legged locomotion community. This letter presents a formal foundation to systematically decompose QP-based centralized nonlinear controllers into a network of lower-dimensional local QPs, with application to legged locomotion. The proposed approach formulates a feedback structure between the local QPs and assumes a one-step communication delay protocol. The properties of local QPs are analyzed, wherein it is established that their steady-state solutions on periodic orbits (representing gaits) coincide with that of the centralized QP. The asymptotic convergence of local QPs' solutions to the steady-state solution is studied via Floquet theory. The effectiveness of the analytical results is evaluated through rigorous numerical simulations and various experiments on a quadrupedal robot, with the result being robust locomotion on different terrains and in the presence of external disturbances. This letter shows that the proposed distributed QPs have considerably less computation time and reduced noise propagation sensitivity than the centralized QP.**

***Index Terms*—Distributed control, robotics, stability of nonlinear systems.**

I. Introduction

QUADRATIC programming (QP)-based controllers have received extensive consideration in recent years for the real-time planning and control of legged robots. For example, they have been used in the context of model predictive control (MPC) for path planning of reduced-order locomotion models (see e.g., [1]–[3]) and whole-body motion control of full-order models (see e.g., [4]–[6]). The bedrock of these approaches considers the *centrality* structure. Despite the superior performance of centralized QP-based controllers, they may exhibit a lack of scalability with the increasing complexity of modern legged robots with higher degrees of freedom (DOFs). In addition, centralized QP-based algorithms cannot be easily transferred into the control of powered prosthetic legs and lower-limb exoskeletons because of the presence of inherent decentralization in the human-robot structure. The curse of dimensionality is a common problem in the field of large-scale systems such as power networks and urban traffic networks [7]. As a solution, a vast majority of work has focused on developing distributed and decentralized control algorithms in large-scale systems, see, e.g., [8]–[11].

In recent years, there have been significant theoretical and numerical advances in developing efficient methods that consider a *distributed* treatment of QPs. Examples include, but are not limited to, active-set methods [12], [13], Lagrangian decomposition or dual decomposition [14], and distributed multiple shooting methods [15]. These methods, however, depend upon the sparse nature of the equality constraints [16]. They translate very well to the problem of network systems or distributed MPC [17] as the subsystems are weakly coupled, hence preserving sparsity. On the contrary, legged robots are underactuated, high-dimensional, and inherently unstable hybrid dynamical systems. By nature, any morphological consideration of subsystems will result in a strong coupling amongst the local systems. This *motivates* the development of distributed QPs for high-DOF legged robots with strong coupling amongst their subsystems.

Steady-state dynamic gaits correspond to periodic trajectories of the hybrid models of locomotion. Different nonlinear control approaches have been developed to stabilize the periodic gaits of these systems. These approaches include but are not limited to hybrid zero dynamics (HZD) [18]–[22], transverse linearization [23], and controlled symmetries [24]. The HZD approach has been integrated with the control Lyapunov functions (CLFs) [4], [5], [25], [26] and control barrier functions (CBFs) [27] to formulate QP-based nonlinear controllers for legged robots. The HZD approach has also been used to synthesize distributed nonlinear controllers for bipedal

Manuscript received February 15, 2022; accepted April 1, 2022. Date of publication April 15, 2022; date of current version April 25, 2022. The work of Vinay R. Kamidi and Kaveh Akbari Hamed was supported by the National Science Foundation (NSF) under Grant 1923216. The work of Jeeseop Kim, Randall T. Fawcett, and Aaron D. Ames was supported by NSF under Grant 1924617, Grant 2128948, and Grant 1923239. Recommended by Senior Editor F. Dabbene. *(Corresponding author: Kaveh Akbari Hamed.)*

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Digital Object Identifier 10.1109/LCSYS.2022.3167795

and quadrupedal robots [26], [28], prosthetic legs [29], [30], and exoskeletons [31]. References [28], [32], [33] developed decentralized controllers based on bilinear matrix inequalities (BMIs) and decomposition for locomotion models. Reference [26] presented stability guarantees, based on CLFs and input-to-state-stability (ISS), for an interconnected system of bipeds. It then developed model-free local QPs for quadrupedal locomotion.

The *overarching goal* of this letter is to present a formal foundation to systematically decompose QP-based and centralized nonlinear controllers into a network of lower-dimensional and local QPs for periodic behaviors in hybrid systems, with application to robotic locomotion. The *objectives* and *key contributions* of this letter are as follows. This letter develops a network of distributed QPs with an inherent feedback structure that preserves the steady-state solution of the centralized QP on periodic orbits. This letter theoretically establishes a set of sufficient conditions under which the solutions of the local QPs asymptotically converge to the steady-state solution of the centralized QP. This letter applies the proposed synthesis approach utilizing local QPs to experimentally realize robust locomotion on the 18-DOF A1 quadrupedal robot. In particular, the theoretical contributions are verified via an extensive set of numerical and experimental studies for quadrupedal locomotion at varying speeds and in the presence of different uncertainties. It is shown that the proposed local QPs have less sensitivity to noise propagation compared to the original centralized QP. It is further shown that the total solve time of the local QPs is reduced by 3.82 times in comparison to the centralized QP.

II. Problem Formulation

The objective of this section is to present the problem statement to synthesize distributed QP-based nonlinear controllers for periodic orbits of hybrid models of locomotion. Without loss of generality, we consider single-domain hybrid systems as follows:

$$\Sigma\colon \begin{cases} \dot{x} = f(x) + g(x)\,u, & x \in \mathcal{X} \\ x^+ = \Delta(x^-), & x^- \in \mathcal{X} \cap \mathcal{S}, \end{cases} \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathcal{U} \subset \mathbb{R}^m$ denote the states and control inputs, respectively, for some positive integers n and m. In addition, $\mathcal{X}$ and $\mathcal{U}$ represent the state manifold and a convex polytope of admissible control inputs. The evolution of the system during continuous-time domains is described by the smooth (i.e., $\mathcal{C}^\infty$) dynamics $\dot{x} = f(x) + g(x)\,u$. In addition, $\mathcal{S}$ denotes the guard of the hybrid system on which the evolution of the system is described by a smooth discrete-time transition (i.e., reset law) as $x^+ = \Delta(x^-)$. Here, x^- and x^+ denote the state of the system right before and after the reset law, respectively.

In this letter, we consider the following class of real-time centralized QPs for nonlinear control of legged locomotion that can arise from input-output (I-O) linearization:

$$\min_{u} \ \frac{1}{2}\sum_{i\in\mathcal{I}} u_i^\top P_{ii}\, u_i \tag{2}$$

$$\text{s.t.} \ \sum_{j\in\mathcal{I}} A_{ij}(x)\, u_j = b_i(x), \quad i \in \mathcal{I} \tag{3}$$

$$u_i \in \mathcal{U}_i,\ i \in \mathcal{I}. \tag{4}$$

Here, we assume that the nonlinear model is composed of $M \geq 2$ interconnected subsystems Σ_i for $i \in \mathcal{I} := \{1, \ldots, M\}$. The local control inputs of the i-th subsystem is given by $u_i \in \mathcal{U}_i \subset \mathbb{R}^{m_i}$. In particular, $u := \mathrm{col}(u_i \,|\, i \in \mathcal{I})$ and we assume that $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_M$ for some convex polytopes $\mathcal{U}_i$ with $\sum_{i\in\mathcal{I}} m_i = m$. In addition, (3) represents the corresponding decomposition for some state-dependent coupled equality constraints given by $A(x)\,u = b(x)$. Furthermore, (4) denotes the decoupled inequality constraints corresponding to feasibility conditions. Finally, P_{ii}'s for $i \in \mathcal{I}$ are positive definite matrices. Throughout this letter, we shall assume that by employing the centralized QP-based controller (2)-(4), there is an asymptotically stable periodic orbit (i.e., gait) $\mathcal{O}$ for the hybrid system model (1).

Assumption 1 (Periodic Solution): We suppose that there is a unique and periodic optimal solution of the strictly convex QP (2)-(4) on the orbit $\mathcal{O}$. This solution, referred to as the *steady-state solution*, is denoted by $u_s^\star(t) := \mathrm{col}(u_{i,s}^\star(t) \,|\, i \in \mathcal{I})$ for $0 \leq t < T$, where T represents the fundamental period of the orbit. It is further assumed that the steady-state solution belongs to the interior of the set $\mathcal{U}_i$, that is, $u_{i,s}^\star(t) \in \mathrm{int}(\mathcal{U}_i)$ for all $i \in \mathcal{I}$ and every $t \in [0, T)$.

Assumption 1 is *not* restrictive in that one can enlarge the admissible set of controls to satisfy the condition $u_{i,s}^\star(t) \in \mathrm{int}(\mathcal{U}_i)$. Throughout this letter, we shall assume that the steady-state solution is known for the proposed network of local QPs. This is *not* a limited assumption as one can consider $u_s^\star(t)$ as the *feedforward control inputs* (e.g., joint-level torques) that generate the orbit $\mathcal{O}$ (i.e., gait). For future purposes, the *steady-state Lagrange multipliers* corresponding to the equality constraints (3) on the periodic orbit are denoted by $\alpha_{i,s}^\star(t)$ for $i \in \mathcal{I}$. The steady-state solution and Lagrange multipliers ($u_s^\star(t)$ and $\alpha_s^\star(t)$) will be used to construct the proposed network of local QPs. Since the QPs will be solved digitally, we will continue our analysis in discrete time. In our notation, $k \in \mathbb{Z}_{\geq 0} := \{0, 1, \ldots,\}$ represents the discrete time, T_s denotes the sampling time, and we assume that $\frac{T}{T_s} = N$ for some positive integer N.

Assumption 2 (One-Step Communication Delay): Motivated by the inherent limitation of the distributed structure, at every time sample k, each local QP can have access to the optimal solution of the other local QPs solved at time $k-1$, but *not* the current time sample. In particular, local QPs can share their optimal solutions from the previous time sample.

Problem 1 (Synthesis of Local QPs): The synthesis problem of distributed controllers consists of designing M local QPs whose optimal solutions asymptotically converge to the steady-state optimal solutions of Assumption 1 while meeting the communication protocol of Assumption 2.

III. Network of Distributed QPs

The objective of this section is to propose a network of local QPs that addresses Problem 1.

A. Synthesis of Local QPs

As discussed before, the local QPs can communicate and share their previous optimal solutions. Let us denote the optimal solutions of the QP_i at time sample $k-1$ by $u_i[k-1]$. We now propose the following structure for the local QP_i, $i \in \mathcal{I}$

$$\begin{aligned} \min_{u_i} \quad & \frac{1}{2} u_i^\top P_{ii}\, u_i + \eta_i^\top u_i \\ \text{s.t.} \quad & A_{ii}\, u_i = b_i - \zeta_i \\ & u_i \in \mathcal{U}_i, \end{aligned} \tag{5}$$

where η_i and ζ_i are *feedback terms* defined as follows:

$$\begin{aligned} \eta_i &:= \sum_{j \in \mathcal{I} \setminus \{i\}} A_{ji}^\top \alpha_{j,\mathrm{s}}^\star[k] - \sum_{j \in \mathcal{I}} L_{ij} \Big(u_j[k-1] - u_{j,\mathrm{s}}^\star[k-1] \Big) \\ \zeta_i &:= \sum_{j \in \mathcal{I} \setminus \{i\}} A_{ij}\, u_{j,\mathrm{s}}^\star[k] \\ &\quad - \sum_{j \in \mathcal{I}} A_{ij}\, K_{ij} \Big(u_j[k-1] - u_{j,\mathrm{s}}^\star[k-1] \Big) \end{aligned} \tag{6}$$

with K_{ij} and L_{ij} for $i, j \in \mathcal{I}$ being proper gain matrices to be determined. For future purposes, the elements of this set of gain matrices can be embedded in a *parameters vector* Θ. In (6), "\" represents the set minus. In the proposed structure, the terms $\eta_i^\top u_i$ and ζ_i are added to the cost function and right-hand side of the equality constraints, respectively. These functions consist of feedforward terms $u_{j,\mathrm{s}}^\star[k]$ and $\alpha_{j,\mathrm{s}}^\star[k]$ for $j \in \mathcal{I} \setminus \{i\}$ as well as feedback terms to penalize the deviation from the orbit. In Theorem 1, we will show that the proposed network of local QPs preserves the steady-state solution $u_{\mathrm{s}}^\star[k]$. Theorem 2 will show that the convergence to the steady-state solution will be achieved via a proper selection of the parameters vector Θ.

Theorem 1 (Steady-State Solutions for QPs): Under Assumptions 1-2, $u_{i,\mathrm{s}}^\star[k]$ for $i \in \mathcal{I}$ are unique optimal solutions for the proposed local QPs in (5) on the desired orbit $\mathcal{O}$.

Proof: Let us assume that at time $k-1$, the solutions for the local QPs coincide with $u_{i,\mathrm{s}}^\star[k-1]$ for all $i \in \mathcal{I}$. This assumption reduces η_i and ζ_i to the feedforward terms, i.e., $\eta_i = \sum_{j \in \mathcal{I} \setminus \{i\}} A_{ji}^\top \alpha_{j,\mathrm{s}}^\star[k]$ and $\zeta_i = \sum_{j \in \mathcal{I} \setminus \{i\}} A_{ij}\, u_{j,\mathrm{s}}^\star[k]$. We then show that $u_{i,\mathrm{s}}^\star[k]$ is the unique solution for the local QP_i at time k. We remark that from Assumption 1, $u_i = u_{i,\mathrm{s}}^\star[k]$ satisfies the equality constraint $A_{ii}\, u_i = b_i - \zeta_i$ and the inequality constraint. Hence, it is a feasible solution. We next show that the first and second-order Karush–Kuhn–Tucker (KKT) optimality conditions are met at this point. Since from Assumption 1 the inequality constraints are inactive (i.e., $u_i \in \mathrm{int}(\mathcal{U}_i)$), the Lagrangian for the local QP (5) is reduced to $\mathcal{L}_i := \frac{1}{2} u_i^\top P_{ii}\, u_i + \eta_i^\top u_i + \alpha_i^\top (A_{ii}\, u_i + \zeta_i - b_i)$, where α_i represents the Lagrange multipliers corresponding to the equality constraints of the QP_i with $i \in \mathcal{I}$. The first-order KKT optimality condition then implies that

$$\frac{\partial \mathcal{L}_i}{\partial u_i} = u_i^\top P_{ii} + \sum_{j \in \mathcal{I} \setminus \{i\}} \alpha_{j,\mathrm{s}}^{\star\top}[k]\, A_{ji} + \alpha_i^\top A_{ii} = 0. \tag{7}$$

It can be shown that this latter equation coincides with the one obtained from the first-order KKT condition for the Lagrangian of the centralized QP (2)-(4). In addition, $\frac{\partial^2 \mathcal{L}_i}{\partial u_i^2} = P_{ii}$ is positive definite, and hence, $u_i = u_{i,\mathrm{s}}^\star[k]$ is indeed the unique optimal solution. ∎

B. Asymptotic Convergence Analysis

From Theorem 1, the local QPs preserve the steady-state solution. We next study the properties and convergence behavior for the solutions of the proposed local QPs. Let us assume that the system's state evolves on the periodic orbit. According to the feedback structure, the optimal solutions of the local QPs at time $k+1$ depend on the ones from time k. In particular, there is a nonlinear and time-varying function $\mathcal{F}$ that defines the following discrete-time dynamics

$$u[k+1] = \mathcal{F}(k, u[k], \Theta), \quad k = 0, 1, \ldots \tag{8}$$

According to the construction procedure, $\mathcal{F}$ is periodic in k with period $N > 0$. In addition, for every parameters vector Θ, $u_{\mathrm{s}}^\star[k] = \mathrm{col}(u_{i,\mathrm{s}}^\star[k] \,|\, i \in \mathcal{I})$ is an N-periodic solution to (8). Next, we make the following assumption.

Assumption 3: The matrices A_{ii}, $i \in \mathcal{I}$ in (3) are full row rank on the periodic orbit $\mathcal{O}$.

Now, we are in a position to present the following result.

Theorem 2 (Local Asymptotic Stability): Under Assumptions 1-3, the following statements hold.

1) The function $\mathcal{F}(k, u, \Theta)$ is continuously differentiable (i.e., $\mathcal{C}^1$) with respect to u on $u = u_{\mathrm{s}}^\star[k]$.
2) The N-periodic solution $u_{\mathrm{s}}^\star[k]$ is locally asymptotically stable for (8) if the eigenvalues of the monodromy matrix lie inside the unit circle, where the monodromy matrix is defined as follows:

$$\Psi_k(\Theta) := \prod_{\ell=0}^{N-1} \frac{\partial \mathcal{F}}{\partial u}(k+\ell, u_{\mathrm{s}}^\star(k+\ell), \Theta), \tag{9}$$

in which the matrices in the product are ordered from right to left for increasing indices ℓ.

Proof: The QPs in (5) can be considered as optimization problems whose cost and constraints are parameterized by (η_i, ζ_i). We aim to show that the solutions of these QPs are $\mathcal{C}^1$ with respect to (η_i, ζ_i) on $u = u_{\mathrm{s}}^\star[k]$. From Theorem 1, $u_{i,\mathrm{s}}^\star[k]$ is the unique optimal solution for the local QPs. The cost and constraints are smooth (i.e., $\mathcal{C}^\infty$) in (u_i, η_i, ζ_i). Since the inequality constraints are inactive at $u_i = u_{i,\mathrm{s}}^\star[k]$, the active constraints are reduced to the equality ones. From Assumption 3, the gradients of the equality constraints with respect to the decision variables are full rank (i.e., regularity condition). Hence, the sufficient conditions of Fiacco's Theorem [34, Th. 2.1] are satisfied. This guarantees the existence, uniqueness, and $\mathcal{C}^1$ continuity of the solutions with respect to (η_i, ζ_i) on an open neighborhood of $u = u_{\mathrm{s}}^\star[k]$. The fact that (η_i, ζ_i) is smooth in terms of u completes the proof of Part 1. Part 2 is an immediate result of Floquet stability theory [35] for periodic and nonlinear discrete-time systems. ∎

IV. Application to Quadrupedal Locomotion

This section aims to numerically and experimentally validate the distributed QP-based controllers for blind quadrupedal locomotion. We consider the 12.45 (kg) A1 quadruped robot (see Fig. 1). The evolution of the robot can be described by 18 DOFs. Here, 6 unactuated DOFs are attributed to the absolute position and orientation of the body. The other 12 DOFs are actuated and associated with the legs' motions: 2-DOFs capture the hip pitch and hip roll motions, and one additional

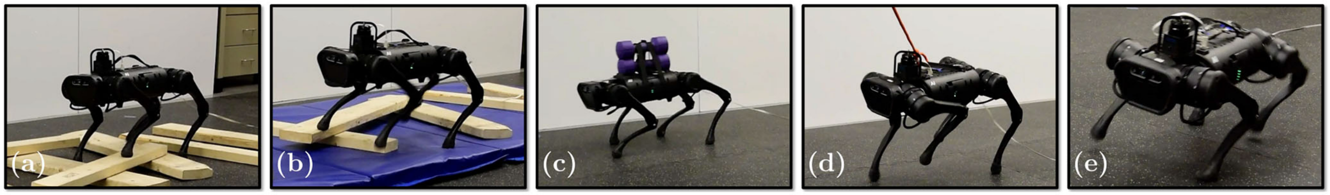

Fig. 1. Snapshots illustrating the performance of the distributed QP-based controllers for a series of blind locomotion experiments (indoor) on the quadrupedal platform, A1. (a) Forward locomotion in a field of arbitrarily dispersed wooden blocks at 0.5 (m/s), (b) trotting on a compliant surface (gym mat) scattered with wooden blocks at 0.4 (m/s), (c) forward locomotion with a payload of 4.54 (kg) (%36 uncertainty) at 0.5 (m/s), (d) tethered pulling while trotting in-place, and (e) stabilization in the presence of external disturbances. Videos of all experiments are available *online* [36].

DOF captures the knee pitch motion for each leg. In this letter, we study double-domain trotting gaits.

A. Hybrid Model of Locomotion

The evolution of the mechanical system during continuous-time domains of locomotion can be described by

$$D(q)\,\ddot{q} + H(q,\dot{q}) = B\,\tau + J^{\top}(q)\,\lambda, \tag{10}$$

where $q \in \mathcal{Q}$ represents the generalized coordinates, $\mathcal{Q} \subset \mathbb{R}^{18}$ denotes the configuration space, $\tau \in \mathcal{T}$ denotes the joint-level torques, $\mathcal{T} \subset \mathbb{R}^{12}$ represents a closed and convex set of admissible torques, $\lambda := \mathrm{col}\{\lambda_\ell \,|\, \ell \in \mathcal{G}\} \in \mathbb{R}^6$ denotes a vector including individual ground reaction fores (GRFs), and $\mathcal{G}$ represents the set of contacting leg ends with the environment. In addition, $D(q) \in \mathbb{R}^{18\times 18}$ represents the mass-inertia matrix, $H(q,\dot{q}) \in \mathbb{R}^{18}$ includes the Coriolis, centrifugal, and gravitational terms, and $B \in \mathbb{R}^{18\times 12}$ is the input matrix. We remark that this model is valid if the GRFs at contacting leg ends remain in the linearized friction cone, denoted by $\mathcal{FC}$, that is, $\lambda_\ell \in \mathcal{FC}$ for all $\ell \in \mathcal{G}$. By defining the state vector as $x := \mathrm{col}(q,\dot{q}) \in \mathcal{X} := \mathcal{Q} \times \mathbb{R}^{18}$, continuous-time dynamics can be described by the following state equation

$$\dot{x} = f(x) + g(x)\,\tau + w(x)\,\lambda. \tag{11}$$

The continuous-time state equation (11) is different from the one in (1). However, the decision variables for the control problem in Section IV-B will contain τ and λ allowing us to represent the QP in the form of (2)-(4). If a leg contacts the ground, the system's state then undergoes a discrete-time transition as in (1) and according to rigid impact models [18].

B. QP-Based I-O Linearizing Controllers

We consider a set of holonomic output functions, referred to as *virtual constraints* [18], to be regulated for the whole-body motion control as follows:

$$y(t,q) := h_0(q) - h_d(t), \tag{12}$$

where $h_0(q)$ encodes a set of controlled variables, and $h_d(t)$ denotes the desired evolution of the controlled variables on the gait $\mathcal{O}$. In this letter, we consider the *front-hind* decomposition for the synthesis of distributed QPs (i.e., $\mathcal{I} = \{1, 2\}$). The output function y for the entire system is heuristically chosen such that it is uniquely separable for two distinct subsystems. We will show that this choice enables us to satisfy Assumption 3. In particular, we choose an 18-dimensional output function for the centralized QP-based controller. More specifically, 6 outputs are designated for the Cartesian coordinates of the center of masses (COMs) for the front and hind subsystems. The remaining 6 outputs are redundant and reserved for the absolute orientations of the COMs. The last 6 outputs are reserved to control the Cartesian positions of two swing leg ends. To address the redundancy in orientation and to ensure the validity of Assumption 3, we introduce defect variables v to be used later in (13). For each distributed QP, the local output $y_i \in \mathbb{R}^9$, $i \in \mathcal{I}$ has 3 components for the corresponding swing leg end and 6 components associated with the position and orientation of the COM. The local torques, $\tau_i \in \mathbb{R}^6$, $i \in \mathcal{I}$, are similarly separated. Next, differentiating the output y along the dynamics (11) gives

$$\ddot{y} = \mathrm{L}_g\mathrm{L}_f y\,\tau + \mathrm{L}_w\mathrm{L}_f y\,\lambda + \mathrm{L}_f^2 y - \ddot{h}_d = -K_P\,y - K_D\,\dot{y} + v, \tag{13}$$

where "L" denotes the Lie derivative, $\mathrm{L}_g\mathrm{L}_f y(t,x)$ and $\mathrm{L}_w\mathrm{L}_f y(t,x)$ represent the decoupling matrices with respect to τ and λ, $v \in \mathbb{R}^{18}$ is the defect variable, and $K_P = 400$ and $K_D = 40$. The objective is to solve for $(\tau,\lambda,v) \in \mathbb{R}^{36}$ that satisfies (13) and contact equations while having feasible torques and GRFs. Contact equations are a set of affine conditions in terms of (τ,λ) that represent the zero acceleration for the stance leg ends. In particular, we have

$$\ddot{p} = \mathrm{L}_g\mathrm{L}_f p\,\tau + \mathrm{L}_w\mathrm{L}_f p\,\lambda + \mathrm{L}_f^2 p = 0, \tag{14}$$

where $p := \mathrm{col}\{p_\ell | \ell \in \mathcal{G}\} \in \mathbb{R}^6$ represents a vector containing the Cartesian coordinates of two stance leg ends.

To solve for a feasible (τ,λ,v), we set up the following real-time and centralized convex QP (whole-body controller)

$$\begin{aligned}
\min_{(\tau,\lambda,v)}\quad & \frac{\gamma_1}{2}\|\tau\|^2 + \frac{\gamma_2}{2}\|\lambda\|^2 + \frac{\gamma_3}{2}\|v\|^2 \\
\text{s.t.}\quad & \mathrm{L}_g\mathrm{L}_f y\,\tau + \mathrm{L}_w\mathrm{L}_f y\,\lambda + \mathrm{L}_f^2 y - \ddot{h}_d = -K_P\,y - K_D\,\dot{y} + v \\
& \mathrm{L}_g\mathrm{L}_f p\,\tau + \mathrm{L}_w\mathrm{L}_f p\,\lambda + \mathrm{L}_f^2 p = 0 \\
& \tau \in \mathcal{T}, \quad \lambda_\ell \in \mathcal{FC}, \quad \forall \ell \in \mathcal{G},
\end{aligned} \tag{15}$$

where $\gamma_1 = 100$, $\gamma_2 = 1$, and $\gamma_3 = 10^7$. To minimize the effect of the defect variable v in the output dynamics, we add a penalty term to the cost function as $\frac{\gamma_3}{2}\|v\|^2$. We remark that the cost function tries to minimize a weighted sum of 2-norms of τ, λ, and v. Additionally, we note that the decision variables are represented by $u := \mathrm{col}(\tau,\lambda,v) \in \mathbb{R}^{36}$ for the centralized QP. Equivalently, the decision variables for the local QPs can be denoted by $u_i := \mathrm{col}(\tau_i,\lambda_i,v_i) \in \mathbb{R}^{18}$ for $i \in \mathcal{I}$. From (15), we can extract the steady-state solution, $u_s^\star(t)$, that satisfies Assumption 1. We are now adequately equipped to decompose the QP in (15) with 36 decision variables, 24 coupled equality constraints and 70 decoupled inequality constraints into two local QPs, as given in (5), with 18 decision variables, 12 equality constraints and 35 inequality constraints. It can be shown that the gradient of the equality constraints

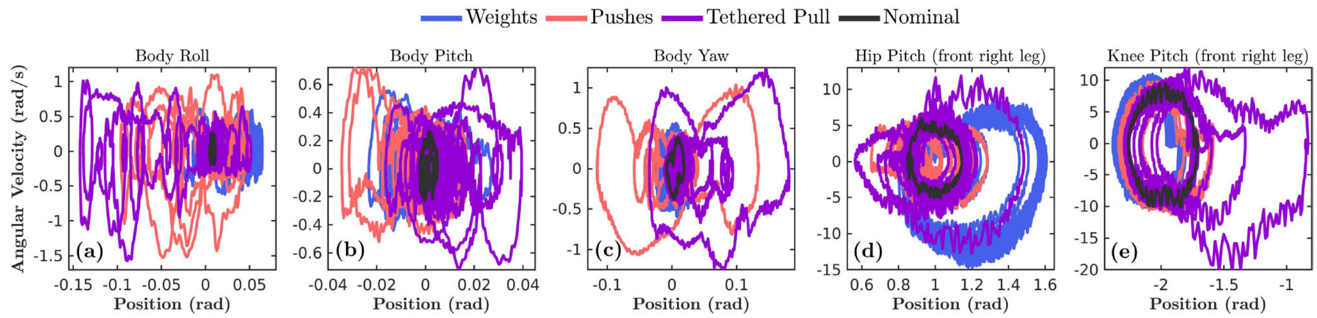

Fig. 2. Phase plots of the unactuated DOFs (roll, pitch, and yaw) and the hip pitch and knee pitch, corresponding to the experiments with payloads (see Fig. 1(c)), tethered pulling (see Fig. 1(d)), and external pushes (see Fig. 1(e)). An overlay of the nominal trot at 0.5 (m/s) is also provided.

with respect to (τ, λ, ν) is full rank as $\mathrm{L}_w \mathrm{L}_f p$ is invertible. This, together with the proper choice of local outputs, meets Assumption 3.

The gains in the structure (6) for local QPs are then tuned as $L_{11} = L_{22} = 10$, $L_{12} = L_{21} = 50$, $K_{11} = K_{22} = 0.08$, and $K_{12} = K_{21} = 0.05$. We employ the qpSWIFT solver [37] at 1 kHz to solve both centralized and local QPs on an off-board laptop equipped with an i7-1185G7 processor running at 3.00 GHz. We note that the centralized and distributed QPs take on average 0.130 (ms) and 0.034 (ms), respectively. In particular, the total solve time for the proposed local QPs is *reduced* by a factor of 3.82. The time-varying Jacobian matrices in (9) are then numerically computed via finite difference and their spectral radius approximately becomes 0.13. This makes the spectral radius of the monodromy matrix over $N = 400$ samples almost zero which indicates the asymptotic convergence to the steady-state solution. Finally, these gains are validated numerically in a physics engine that considers the hybrid nature of locomotion.

C. Numerical and Experimental Evaluation

The distributed QP-based controller (5) is first validated through extensive numerical simulations in RaiSim [38]. Subsequently, we experimentally evaluate the performance of the local QPs on the A1 platform while subjecting the robot to various uncertainties. Specifically, we consider (a) a terrain arbitrarily dispersed with wooden blocks (see Fig. 1(a)), (b) a compliant surface (gym mat) arbitrarily scattered with wooden blocks (see Fig. 1(b)), and (c) locomotion with a payload of 4.54 (kg) (36% uncertainty in the total mass) (see Fig. 1 (c)). We consider a speed of 0.5 (m/s) for experiments in (a) and (c) and a speed of 0.4 (m/s) for the one in (b). Additionally, we consider (d) tethered pulling (see Fig. 1(d)) and (e) external pushes (see Fig. 1(e)) for in-place trot locomotion. The proposed local controllers can robustly stabilize the gaits, as evident from the phase plots in Fig. 2. Especially, Fig. 2(c) illustrates the local controller's far-reaching capabilities by demonstrating convergence to the nominal orbit even after experiencing aggressive pulls and pushes. Videos of all experiments are available *online* [36].

Figure 3(a) depicts the experimental torque of the hip roll from the distributed QP in contrast to the corresponding steady-state torque (extracted from numerical simulations) and highlights the close convergence of the local torque to the prescribed one. The small error between the steady-state and distributed torques can be attributed to the model uncertainty and measurement noise from hardware experimentation. We remark that as each QP knows the solutions from the other QP with a one-step communication delay, the solution is sub-optimal compared to the centralized QP. However, as observed from Fig. 3(a), this solution ultimately converges to the steady-state solution as shown theoretically. Furthermore, our experiments indicate that the local solutions have almost the same robustness as the centralized one for robot locomotion. To compare the performance of centralized and distributed QPs, we inject noise into the velocity components of the front subsystem (in numerical simulations) with a signal-to-noise ratio (SNR) of 1 (dB). As a result, we notice noisy torques in the front subsystem under both the control schemes. However, for the hind subsystem, noise propagates only with the centralized QP-based controller. The distributed QP-based controller significantly rejects the noise (see Fig. 3(b)). In particular, the computed SNR of the torque signal for one stance domain is increased from 18.35 (dB) in the centralized QP to 37.37 (dB) in the distributed QP. This property can be primarily related to the fact that each subsystem deals with its own Lie derivatives and decoupling matrices arising from (15) in the structure of (5).

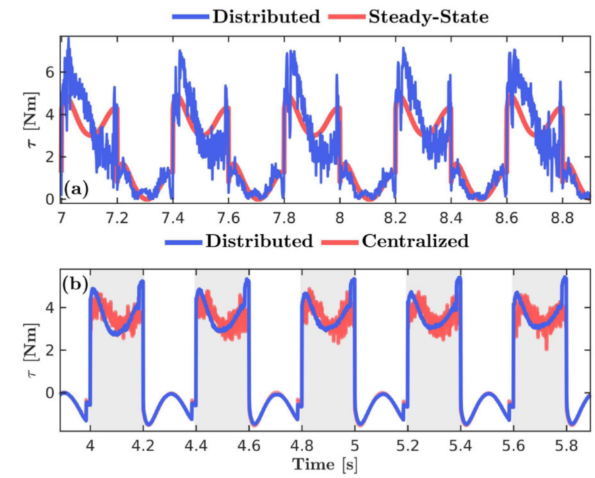

Fig. 3. (a) Comparison of the hip roll torque of the front right leg against the corresponding steady-state torque during nominal trot experiment, demonstrating close convergence. (b) Illustration of the simulated hip roll torque of the hind right leg in the presence of noisy velocities in the front subsystem. Here, the gray overlay indicates ground contact.

V. Conclusion

This letter presented a formal foundation to decompose QP-based nonlinear controllers into a network of low-dimensional and distributed QP-based controllers with an inherent feedback structure for the general class of legged locomotion models. The proposed network of local QPs is developed based on a one-step communication delay protocol and preserves the steady-state solution of the centralized QP on periodic orbits. Properties of the local QPs were studied to establish a set of sufficient conditions under which the solutions of the local QPs asymptotically converge to the steady-state solution of the centralized QP. The theoretical results were then applied for the synthesis of distributed I-O linearizing controllers for realizing agile quadrupedal locomotion. In particular, the analytical results' effectiveness was verified via rigorous numerical and experimental studies for the blind and robust locomotion of the A1 quadrupedal robot in the presence of disturbances and terrain uncertainties. It is shown that the total solve time of the local QPs is reduced by 3.82 times in comparison to the centralized QP. Additionally, a simulated case study demonstrated that favoring the proposed distributed QP structure presents better noise-rejection properties.

For future work, we will examine the performance of distributed QPs while considering a larger number of subsystems (e.g., each leg of the robot). Evaluation of the proposed controller beyond trot gaits will also be a topic of consideration.

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