Lyapunov-Like Conditions for the Existence of Zeno Behavior in Hybrid and Lagrangian Hybrid Systems

Andrew Lamperski and Aaron D. Ames

***Abstract*— Lyapunov-like conditions that utilize generalizations of energy and barrier functions certifying Zeno behavior near Zeno equilibria are presented. To better illustrate these conditions, we will study them in the context of Lagrangian hybrid systems. Through the observation that Lagrangian hybrid systems with *isolated* Zeno equilibria must have a one-dimensional configuration space, we utilize our Lyapunov-like conditions to obtain easily verifiable necessary and sufficient conditions for the existence of Zeno behavior in systems of this form.**

I. Introduction

Zeno behavior occurs in a hybrid system if an infinite number of discrete transitions occur in a finite amount of time. The goal of this paper is to study a specific type of Zeno behavior, namely, Zeno behavior that occurs near *isolated Zeno equilibria*. A Zeno equilibrium is a set of points in the continuous domains of a hybrid system that are invariant under the discrete dynamics (but not the continuous dynamics). Isolated Zeno equilibria form the Zeno analogue of isolated equilibria. Just as there can be complicated sets of equilibria in dynamical systems, there can be much more complicated sets of Zeno equilibria. Yet as the study of isolated equilibria of dynamical systems has been hugely successful, we believe that an important first step to the understanding of Zeno behavior is the detailed study of isolated Zeno equilibria.

The two main results of this paper are: Lyapunov-like sufficient conditions for existence of Zeno behavior near isolated equilibria, and a characterization of Zeno behavior near isolated Zeno equilibria of Lagrangian hybrid systems.

Unlike classical Lyapunov theory, our Lyapunov-like conditions for the existence of Zeno behavior actually involve two functions on each domain. The first function is used in a manner similar to standard Lyapunov functions, i.e., it is used to prove positive invariance of compact sets around our Zeno equilibria. The second function is a barrier function [1] used to capture explicit information about how long an execution spends in a continuous domain. In addition, these functions appear to be searchable via polynomial optimization [2], [3]. Therefore, this result works toward the goal of automated analysis of hybrid systems.

The second main result presented in this paper gives necessary and sufficient conditions for Zeno behavior near isolated Zeno equilibria of Lagrangian hybrid systems, which model mechanical systems undergoing impacts. To obtain these conditions, we first show that Lagrangian hybrid systems with isolated Zeno equilibria must necessarily have a one-dimensional configuration space. This simple structure allows us to explicitly construct the desired Lyapunov-like functions that prove the existence of Zeno behavior. Moreover, due to the explicit nature of these functions, we are able to demonstrate that one can check for the existence and nonexistence of Zeno behavior by evaluating functions given in the system description at a single point. This result represents, to the authors' knowledge, the only known necessary and sufficient result for Zeno behavior in uncontrolled hybrid systems. Necessary and sufficent conditions for Zeno behavior in a significantly different class of controlled hybrid systems were found in [4].

Although results presented in this paper make gains toward a better understanding of Zeno behavior, they also indicate possible future research directions. The most compelling of these is motivated by the fact that isolated Zeno equilibria only occur in Lagrangian hybrid systems with one-dimensional configuration spaces, indicating that to understand Zeno behavior in Lagrangian hybrid systems, more complicated sets of Zeno equilibria must be studied. Also, due to the Lyapunov-like nature of the sufficient conditions presented, the hope is that in the future these conditions could be automated. Finally, the Lyapunov-like conditions given in this paper also imply that Zeno executions converge to the Zeno equilibria; that is, Zeno behavior is closely related to the stability of Zeno equilibria. This relationship was first noticed in [5], where the *asymptotic* stability of Zeno equilibria was studied; in fact, the Lyapunov-like conditions presented in that work are very similar to the conditions presented in this paper. Yet, the claim made in [5] that the asymptotic stability of Zeno equilibria implies the existence of Zeno behavior has since proven to be incorrect. Therefore, the connection between the stability of Zeno equilibria and Zeno behavior still remains to be made, although this paper provides strong evidence for the existence of such a connection.

Due to the subtle and complex nature of Zeno behavior, it has been studied in many forms and from many different perspectives. Most of the conditions for Zeno behavior are necessary and tend to be very conservative; see [6], [7], [8] for general hybrid systems, and [9], [10] for linear complementarity systems. Obtaining sufficient conditions for the existence of Zeno behavior seems to be a much more challenging task. Therefore, very few such conditions have been given to date and those that have been obtained apply to very limited classes of hybrid systems [11]. We also

A. Lamperski and A. D. Ames are with Control and Dynamical Systems Department, California Institute of Technology, Pasadena, CA 91125 {andyl,ames}@cds.caltech.edu

note that this paper studies Zeno behavior in Lagrangian hybrid systems, which were studied in [12], [13] and [14] as motivated by [15]. Finally, the characterization of Zeno behavior presented in this paper complements the topological characterization of Zeno behavior presented in [16].

II. Hybrid Systems & Zeno Equilibria

In this section, we introduce the basic notations on which the rest of the paper will build. That is, we define hybrid systems, executions, Zeno equilibria and convergence to Zeno equilibria.

Definition 1: A *hybrid system on a cycle* is a tuple:

$$\mathscr{H} = (\Gamma, D, G, R, F),$$

where

- $\Gamma = (Q, E)$ is a directed cycle, with

$$\begin{aligned} Q &= \{q_0, \ldots, q_{k-1}\}, \\ E &= \{e_0 = (q_0, q_1), e_1 = (q_1, q_2), \\ &\qquad \ldots, e_{k-1} = (q_{k-1}, q_0)\}. \end{aligned}$$

  We denote the source of an edge $e \in E$ by $\mathrm{source}(e)$ and the target of an edge by $\mathrm{target}(e)$.
- $D = \{D_q\}_{q\in Q}$ is a set of *domains*, where D_q is a smooth manifold.
- $G = \{G_e\}_{e\in E}$ is a set of *guards*, where $G_e \subseteq D_{\mathrm{source}(e)}$ is and embedded submanifold of $D_{\mathrm{source}(e)}$.
- $R = \{R_e\}_{e\in E}$ is a set of *reset maps*, where $R_e : G_e \subseteq D_{\mathrm{source}(e)} \to D_{\mathrm{target}(e)}$ is a smooth map.
- $F = \{f_q\}_{q\in Q}$, where $f_q : D_q \to TD_q$ is a Lipschitz vector field on D_q.

Remark 1: Note that the motivation for considering hybrid systems on cycles, rather than hybrid systems on general graphs, is that cycles are indicative of Zeno behavior (see [6] and [8]). Therefore, beginning with hybrid systems defined on cycles will greatly simplify our analysis, while still capturing characteristic types of Zeno behavior.

Definition 2: An *execution* of a hybrid system $\mathscr{H} = (\Gamma, D, G, R, F)$ is a tuple:

$$\chi = (\Lambda, I, \rho, C)$$

where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is a finite or infinite indexing set,
- $I = \{I_i\}_{i\in\Lambda}$ where for each $i \in \Lambda$, I_i is defined as follows: $I_i = [\tau_i, \tau_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \tau_N)$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, N finite. Here, for all $i, i+1 \in \Lambda$, $\tau_i \leq \tau_{i+1}$ with $\tau_i, \tau_{i+1} \in \mathbb{R}$, and $\tau_{N-1} \leq \tau_N$ with $\tau_{N-1}, \tau_N \in \mathbb{R}$.
- $\rho : \Lambda \to Q$ is a map such that for all $i, i+1 \in \Lambda$, $(\rho(i), \rho(i+1)) \in E$. This is the *discrete component* of the execution.
- $C = \{c_i\}_{i\in\Lambda}$ is a set of *continuous trajectories*, and they must satisfy $\dot{c}_i(t) = f_{d(i)}(c_i(t))$ for $t \in I_i$.

We require that when i, $i+1 \in \Lambda$,

$$\begin{array}{ll} \text{(i)} & c_i(t) \in D_{\rho(i)} \;\forall\; t \in I_i \\ \text{(ii)} & c_i(\tau_{i+1}) \in G_{(\rho(i),\rho(i+1))} \\ \text{(iii)} & R_{(\rho(i),\rho(i+1))}(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1}). \end{array} \tag{1}$$

When $i = |\Lambda| - 1$, we still require that (i) holds.

Note that the continuous initial condition of an execution χ is given by $c_0(\tau_0) \in D_{\rho(0)}$. The discrete initial condition is given by $\rho(0)$.

The object of study in this paper will be Zeno executions, which are defined in the following manner:

Definition 3: An execution χ is *Zeno* if $\Lambda = \mathbb{N}$ and

$$\lim_{i\to\infty} \tau_i = \tau_0 + \sum_{i=0}^{\infty} \tau_{i+1} - \tau_i = \tau_\infty$$

for some finite $\tau_\infty \in \mathbb{R}$.

A hybrid system $\mathscr{H}$ is *Zeno*[1] if there exists a Zeno execution χ such that $\tau_{i+1} - \tau_i \neq 0$ for some $i \in \mathbb{N}$.

Zeno behavior can be likened to stability, in that both types of behavior involve convergence. This motivates the study of the type of equilibria associated to Zeno behavior: Zeno equilibria. For more on Zeno behavior and Zeno equilibria, see [12].

Definition 4: A *Zeno equilibria* of a hybrid system $\mathscr{H} = (\Gamma, D, G, R, F)$ is a set $z = \{z_q\}_{q\in Q}$ satisfying the following conditions for all $q \in Q$:

- For the unique edge $e = (q, q') \in E$
  - $z_q \in G_e$,
  - $R_e(z_q) = z_{q'}$,
- $f_q(z_q) \neq 0$.

A Zeno equilibrium $z = \{z_q\}_{q\in Q}$ is *isolated* if there are neighborhoods $W_q \subset D_q$, $q \in Q$, such that there does not exist another Zeno equilibrium $\hat{z} = \{\hat{z}_q\}_{q\in Q}$ with $\hat{z}_q \in W_q$ for all $q \in Q$.

Note that, in particular, the conditions given in Definition 4 imply that for all $i \in \{0, \ldots, k-1\}$,

$$R_{e_{i-1}} \circ \cdots \circ R_{e_0} \circ R_{e_{k-1}} \circ \cdots \circ R_{e_i}(z_i) = z_i.$$

That is, the element z_i is a fixed point under the reset maps composed in a cyclic manner.

Definition 5: An execution χ with $\Lambda = \mathbb{N}$ *converges* to a zeno equilibrium $z = \{z_q\}_{q\in Q}$ if given any neighborhoods W_q with $z_q \in W_q$ for all $q \in Q$, there exists $j \in \Lambda$ such that $c_k(t) \in W_{\rho(k)}$ for all $k \geq j$ and all $t \in I_k$.

III. Lyapunov-Like Conditions for the Existence of Zeno Behavior

This section presents Lyapunov-like sufficient conditions for the existence of Zeno behavior in hybrid systems on cycles. These results apply to a general class of hybrid

[1]The motivation for this definition is that we want to exclude the possibility that a hybrid system is "trivally" Zeno, i.e., the only Zeno executions are executions that begin at a Zeno Equilibria

systems, and are fundamental in establishing the results presented in the Section V.

Conditions for Zeno behavior. We begin by introducing the conditions that will imply the existence of Zeno behavior. These involve conditions on the continuous and discrete portion of the hybrid system, together with the "convergence conditions" that indicate how executions will tend toward Zeno equilibria. As these conditions are, in some sense, analogous to stability conditions, it is not surprising that they involve Lyapunov-like functions. In fact, on each domain, they utilize two such functions—one is analogous to a standard Lyapunov function, while the other is analogous to a Barrier function [1]—that will interact through the "convergence conditions." Our conditions closely resemble the conditions given in [5] but avoid problems arising from slow convergence by enforcing geometric convergence in **C**1.

Let $\{z_q\}_{q\in Q}$ be a Zeno equilibrium of a hybrid system $\mathscr{H}$, $z_q \in W_q \subset D_q$ be open neighborhoods of z_q for all $q \in Q$, $V_q, B_q \in C^1(W_q, \mathbb{R})$ and $r < 1$, $a > 0$ and $b > 0$ be constants. With this notation, consider the following conditions:

Continuous Conditions for Events:

- **EC**1: $V_q(x) > 0$ for all $x \in W_q \setminus \{z_q\}$ and $V_q(z_q) = 0$ for all $q \in Q$.
- **EC**2: $\dot{V}_q(x) = dV_q(x) \cdot f_q(x) \leq 0$ for all $x \in W_q$ and $q \in Q$.
- **EC**3: $B_q(x) \geq 0$ for all $x \in W_q$ and $q \in Q$.
- **EC**4: $\dot{B}_q(x) = dB_q(x) \cdot f_q(x) < 0$ for all $x \in W_q$, and $q \in Q$.

Discrete Condition for Events:

- **ED**1: $V_{q'}(R_e(x)) \leq V_q(x)$ for all $x \in G_e \cap W_q$, $q \in Q$ and the unique edge $e = (q, q')$.

Convergence Conditions:

- **C**1: There exists $q \in Q$ such that $V_{q'}(R_e(x)) \leq rV_q(x)$ for all $x \in G_e \cap W_q$.
- **C**2: $B_q(x) \leq b(V_q(x))^a$ for all $x \in R_{\hat{e}}(G_{\hat{e}}) \cap W_q$, $q \in Q$ and the unique edge $\hat{e} = (q', q)$.

With these conditions in hand, we now present the main result of this section.

Theorem 1: *Let $\{z_q\}_{q\in Q}$ be an isolated Zeno equilibrium of a hybrid system $\mathscr{H}$. If conditions* **EC***1-***EC***4,* **ED***1,* **C***1 and* **C***2 hold, then there exists a neighborhood Z_q of z_q for all $q \in Q$ such that for all $x_0 \in Z_q$ there exists a Zeno execution $\chi = (\Lambda, I, \rho, C)$ with $c_0(\tau_0) = x_0$. Therefore, $\mathscr{H}$ is Zeno.*

We prove this theorem using two lemmas.

Lemma 1: *Let $x(t)$ be the solution to $\dot{x} = f_q(x)$. If conditions* **EC***1-***EC***4 hold, then there exists a compact set $\Omega_q^\alpha \subset W_q$, indexed by $\alpha > 0$, and a positive constant γ_q with the following properties:*

- *There exists an open set Z_q such that $z_q \in Z_q \subset \Omega_q^\alpha$.*
- *There exists a time*

$$\tau \leq \frac{B_q(x(0))}{\gamma_q} < \infty$$

such that $x(\tau) \in G_e$ when $x(0) \in \Omega_q^\alpha$, where here $e = (q, q')$.

Proof: Similar to Lyapunov's theorem, **EC**1 implies that when $\alpha > 0$ is small enough, the connected component of the set:

$$S_q^\alpha = \{x \in W_q : V_q(x) \leq \alpha\} \tag{2}$$

containing z_q is compact and contains a neighborhood of z_q. So we can restrict W_q to a neighborhood $\bar{W}_q$ so that

$$\Omega_q^\alpha = \{x \in \bar{W}_q : V_q(x) \leq \alpha\} \tag{3}$$

is connected and compact. Further, Ω_q^α contains a neighborhood Z_q of z_q.

Since we have only a finite number of domains, we can pick $\bar{W}_q$ and a single α such that Ω_q^α is connected and compact for each domain.

From **EC**2, if $x(0) \in \Omega_q^\alpha$, then $x(t) \in \Omega_q^\alpha$ for all $t \geq 0$ as long as $x(t) \in D_q$. That is, $x(t)$ remains in Ω_q^α unless an event occurs.

Now we use **EC**3 and **EC**4 to show that events occur in finite time. Let

$$-\gamma_q = \max_{x\in\Omega_q^\alpha} \dot{B}_q(x) \tag{4}$$

Since Ω_q^α is compact and $\dot{B}_q(x) < 0$ for $x \in \Omega_q^\alpha \subset W_q$, we must have $\gamma_q > 0$. So for $t \geq 0$

$$B_q(x(t)) = B_q(x(0)) + \int_0^t \dot{B}_q(x(\sigma))d\sigma \leq B_q(x(0)) - \gamma_q t.$$

Therefore, $x(t)$ must reach the guard within time $t^* = B_q(x(0))/\gamma_q$ by **EC**3. ∎

Lemma 2: *If conditions* **EC***1-***EC***4 and* **ED***1 hold, then for each $x_0 \in \Omega_q^\alpha$, with $\alpha > 0$ sufficiently small, there exists a unique execution $\chi = (\Lambda, I, \rho, C)$ with $c_0(\tau_0) = x_0$, $\rho(0) = q$ and $\Lambda = \mathbb{N}$.*

Proof: Continuity of R_e and (**ED**1) imply that for $\alpha > 0$ as in Lemma 1 and all $e = (q, q') \in E$, $R_e(G_e \cap \Omega_q^\alpha) \subset \Omega_{q'}^\alpha$.

Let $c_0(\tau_0) = x_0 \in \Omega_q^\alpha$. Assume without loss of generality that $q = q_0$. Then, since our hybrid system evolves on a directed cycle, conditions (i)-(iii) in (1) imply that

$$\rho(i) := q_{\bar{i}}, \qquad \bar{i} := i \mod |Q|$$

So by construction, $c_0(\tau_0) \in \Omega_{\rho(0)}^\alpha$.

Now, inductively assume that $c_i(\tau_i) \in \Omega_{\rho(i)}^\alpha$ for some $i \geq 0$. From Lemma 1, we can extend $c_i(\tau_i)$ to a solution $c_i(t)$ of $\dot{x} = f_{\rho(i)}(x)$ such that at some finite time τ_{i+1}, we have $c_i(\tau_{i+1}) \in G_{e_{\bar{i}}} \cap \Omega_{\rho(i)}^\alpha$. From above, $R_{e_{\bar{i}}}(c_i(\tau_{i+1})) \in \Omega_{\rho(i+1)}^\alpha$. Therefore $c_i(\tau_i) \in \Omega_{\rho(i)}^\alpha$ implies that $c_{i+1}(\tau_{i+1}) \in \Omega_{\rho(i+1)}^\alpha$.

The above argument holds inductively, so if $c_0(\tau_0) \in \Omega_{q_0}^\alpha$, then $c_0(\tau_0)$ extends to a unique execution $\chi = (\Lambda, I, \rho, C)$ with $\Lambda = \mathbb{N}$. ∎

We will now use Lemmas 1 and 2 to prove Theorem 1

Proof: [*of Theorem 1*] Take $x_0 \in \Omega_{q_0}^\alpha$. Let χ be the unique infinite execution with $c_0(\tau_0) = x_0 \in \Omega_{q_0}^\alpha$. We show the total time spent in D_{q_0} is finite.

Repeated applications of **C**1 show that

$$V_{q_0}(c_{j|Q|}(\tau_{j|Q|})) \leq r^j V_{q_0}(c_0(\tau_0)). \tag{5}$$

Since $\Lambda = \mathbb{N}$ and $c_{j|Q|}(t) \in W_{q_0}$ for all $j \geq 0$, $t \in I_{j|Q|}$ we can assume without loss of generality that $c_{j|Q|}(\tau_{j|Q|}) \in R_{e_{|Q|}}(G_{e_{|Q|}}) \cap W_{q_0}$. Lemma 1 guarantees that the interval $I_{j|Q|}$ has length $\tau_{j|Q|+1} - \tau_{j|Q|} \leq B_{q_0}(c_{j|Q|}(\tau_{j|Q|}))/\gamma_{q_0}$. Thus we see the total time spent in D_{q_0} is

$$\begin{aligned}\sum_{j=0}^{\infty}(\tau_{j|Q|+1} - \tau_{j|Q|}) &\leq \frac{1}{\gamma_{q_0}}\sum_{j=0}^{\infty} B_{q_0}(c_{j|Q|}(\tau_{j|Q|})) \\ &\leq \frac{b}{\gamma_{q_0}}\left(V_{q_0}(c_0(\tau_0))\right)^a \sum_{j=0}^{\infty}(r^a)^j \\ &< \infty,\end{aligned}$$

where second inequality follows from **C**2 and (5), and the final inequality follows since $r < 1$ and $a > 0$.

The same argument shows that the execution spends only a finite amount of time in domain D_q for all $q \in Q$. ■

Corollary 1: *Any execution χ with $\Lambda = \mathbb{N}$ and $c_0(\tau_0) \in Z_{\rho(0)}$ converges to $\{z_q\}_{q\in Q}$.*

IV. Simple Hybrid Mechanical Systems

Mechanical systems undergoing impacts are naturally modeled as hybrid systems. In this section, we will consider hybrid systems of this form and demonstrate how one obtains such systems from hybrid Lagrangians, which are the hybrid analogue of Lagrangians. For more on hybrid Lagrangians and Lagrangian hybrid systems, see [12], [13] and [14].

Lagrangians. Consider a configuration space[2] Θ and a Lagrangian $L : T\Theta \to \mathbb{R}$ given in coordinates by:

$$L(\theta, \dot\theta) = \frac{1}{2}\dot\theta^T M(\theta)\dot\theta - U(\theta) \tag{6}$$

where $M(\theta)$ is positive definite and symmetric and $U(\theta)$ is the potential energy. For the sake of simplicity, we assume $\Theta \subset \mathbb{R}^n$ since all our results are local, i.e., we can work within a coordinate chart. The equations of motion are then given in coordinates by the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = 0.$$

In the case of Lagrangians of the form given in (6), the Euler-Lagrange equations become:

$$M(\theta)\ddot\theta + C(\theta, \dot\theta)\dot\theta + N(\theta) = 0,$$

where $C(\theta, \dot\theta)$ is the *Coriolis matrix* and $N(\theta) = \frac{\partial U}{\partial \theta}(\theta)$. Setting $x = (\theta^T, \dot\theta^T)^T$, the Lagrangian vector field, f_L, associated to L takes the familiar form

$$\dot x = f_L(x) = \begin{pmatrix} \dot\theta \\ M(\theta)^{-1}(-C(\theta,\dot\theta)\dot\theta - N(\theta)) \end{pmatrix}. \tag{7}$$

This process of associating a dynamical system to a Lagrangian will be mirrored in the setting of hybrid systems. First, we introduce the notion of a hybrid Lagrangian.

[2]Note that we denote the configuration space by Θ rather than Q, due to the fact that Q denotes the vertices of the graph of a hybrid system.

Definition 6: A *hybrid Lagrangian* is a tuple, $\boldsymbol{L} = (\Theta, L, h)$, where

- $\Theta \subset \mathbb{R}^n$ is the configuration space,
- $L : T\Theta \to \mathbb{R}$ is a Lagrangian of the form give in (6),
- $h : \Theta \to \mathbb{R}$ is a *unilateral constraint function*, where we assume that 0 is a regular value of h.

Domains from constraints. Given a smooth (unilateral constraint) function $h : \Theta \to \mathbb{R}$ on a configuration space Θ such that $h^{-1}(0)$ is a smooth manifold, i.e., 0 is a regular value of h, we can construct a domain and a guard explicitly. To this constraint function we have an associated domain, D_h, defined to be the manifold (with boundary):

$$D_h = \{(\theta, \dot\theta) \in T\Theta : h(\theta) \geq 0\}.$$

Similarly, we have an associated guard, G_h, defined as the following submanifold of D_h:

$$G_h = \{(\theta, \dot\theta) \in T\Theta : h(\theta) = 0 \text{ and } dh_\theta\dot\theta \leq 0\},$$

where $dh(\theta) = \begin{pmatrix} \frac{\partial h}{\partial \theta_1}(\theta) & \cdots & \frac{\partial h}{\partial \theta_n}(\theta) \end{pmatrix}$. Note that the requirement that 0 is a regular value of h is equivalent to requiring that $dh(\theta) \neq 0$ when $h(\theta) = 0$.

Lagrangian Hybrid Systems. Given a hybrid Lagrangian $\mathbf{L} = (\Theta, L, h)$, the *Lagrangian hybrid system associated to* $\mathbf{L}$ is the hybrid system

$$\mathscr{H}_{\mathbf{L}} = (\Gamma = (\{q\}, \{(q,q)\}), D_{\mathbf{L}}, f_{\mathbf{L}}, G_{\mathbf{L}}, R_{\mathbf{L}}),$$

where $D_{\mathbf{L}} = \{D_h\}$, $f_{\mathbf{L}} = \{f_L\}$, $G_{\mathbf{L}} = \{G_h\}$ and $R_{\mathbf{L}} = \{R_h\}$ with $R_h(\theta, \dot\theta) = (\theta, P(\theta, \dot\theta))$, where

$$P(\theta, \dot\theta) = \dot\theta - (1+e)\frac{dh(\theta)\dot\theta}{dh(\theta)M(\theta)^{-1}dh(\theta)^T}M(\theta)^{-1}dh(\theta)^T. \tag{8}$$

Example 1: To provide an example of the concepts introduced in this paper, we will consider the hybrid system modeling a pendulum impacting the ground. This system has as its hybrid Lagrangian the tuple $\mathbf{L}_{\text{pend}} = (\mathbb{R}, L_{\text{pend}}, h_{\text{pend}})$, where $L_{\text{pend}}(\theta, \dot\theta) = \frac{1}{2}ml^2\dot\theta^2 - mgl\sin(\theta)$ and $h_{\text{pend}}(\theta) = \sin(\theta)$, with θ the angle of the pendulum from horizontal, m the mass of the bob, l the length of the rod and g the acceleration due to gravity.

From the hybrid Lagrangian $\mathbf{L}_{\text{pend}}$, we obtain the hybrid model of the pendulum:

$$\mathscr{H}_{\text{pend}} = (\Gamma = (\{q\}, \{(q,q)\}), D_{\text{pend}}, G_{\text{pend}}, R_{\text{pend}}, F_{\text{pend}}),$$

where $D_{\text{pend}} = \{D_{h_{\text{pend}}}\}$, $F_{\text{pend}} = \{f_{L_{\text{pend}}}\}$, $G_{\text{pend}} = \{G_{h_{\text{pend}}}\}$ and $R_{\text{pend}} = \{R_{h_{\text{pend}}}\}$ with

$$\begin{aligned} D_{h_{\text{pend}}} &= \left\{\begin{pmatrix}\theta\\ \dot\theta\end{pmatrix} \in \mathbb{R}^2 : \sin(\theta) \geq 0\right\} \\ G_{h_{\text{pend}}} &= \left\{\begin{pmatrix}\theta\\ \dot\theta\end{pmatrix} \in \mathbb{R}^2 : \sin(\theta) = 0 \text{ and } \dot\theta \leq 0\right\} \\ R_{h_{\text{pend}}}(\theta, \dot\theta) &= \begin{pmatrix}\theta\\ -e\dot\theta\end{pmatrix} \\ f_{L_{\text{pend}}}(\theta, \dot\theta) &= \begin{pmatrix}\dot\theta\\ -\frac{g}{l}\cos(\theta)\end{pmatrix}\end{aligned}$$

where here $0 \leq e \leq 1$ is the coefficient of restitution.

We note that $(0,0)$ and $(\pi,0)$ are isolated Zeno equilibria of $\mathscr{H}_{\text{pend}}$. Moreover, it is easy to verify through Theorem 2 that this system is Zeno, although we will use the results presented in the next section to establish this fact. Also note that this hybrid system is actually characteristic of Lagrangian hybrid systems that have isolated Zeno equilibria and are Zeno as will be seen in the next section.

V. Zeno Behavior in Lagrangian Hybrid Systems

In this section, we characterize Zeno behavior in Lagrangian hybrid systems with isolated Zeno equilibria. This characterization relies, in a fundamental fashion, on the following proposition.

Proposition 1: *If the Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$ has an isolated Zeno equilibrium, then*

L1: *Θ is one-dimensional.*

L2: *The Zeno equilibria takes the form*

$$z = \left\{ \begin{pmatrix} \theta^* \\ 0 \end{pmatrix} \right\}.$$

L3: *$P(\theta,\dot{\theta}) = -e\dot{\theta}$, where $0 \leq e \leq 1$ is the coefficient of restitution.*

Proof: **L**1: If $z = \{(\theta^*,\dot{\theta}^*)^T\}$ is an isolated Zeno equilibrium of $\mathscr{H}_L$, then there exists a neighborhood W of $(\theta^*,\dot{\theta}^*)^T$ that contains no equilibria of (7) and no other Zeno equilibria. It follows that that $(\theta^*,\dot{\theta}^*)^T$ is the unique fixed point of R_h in W. Now for $(\theta,\dot{\theta}) \in G_L$, $h(\theta) = 0$ and $dh(\theta) \neq 0$ so using the fact that $M(\theta)^{-1}$ is positive definite, it follows that:

$$\begin{aligned} R_h(\theta,\dot{\theta}) = (\theta,\dot{\theta}) \quad &\Longleftrightarrow \quad (dh(\theta)\dot{\theta})M(\theta)^{-1}dh(\theta)^T = 0 \\ &\Longleftrightarrow \quad dh(\theta)\dot{\theta} = 0. \end{aligned}$$

Therefore, if $\dim(T_\theta\Theta) \geq 2$ then $dh(\theta)\dot{\theta} = 0$ defines a hyperplane in $T_\theta\Theta$ of points in the guard, each of which is a fixed point of R_h. Thus, when $\dim(\Theta) \geq 2$, there can be no isolated Zeno equilibria.

L2 and **L**3 follow from **L**1. ∎

Notation 1: From this point on, we adopt the following notation: $dh(\theta) = h'(\theta)$ and $\frac{\partial U}{\partial \theta}(\theta) = U'(\theta)$, since we will be interested in h and U that are functions of one real variable as a result of Proposition 1.

A Characterization of Zeno Behavior. We have shown that hybrid Lagrangian systems with isolated Zeno equilibria are necessarily quite simple. The main result in this section gives an explicit characterization of isolated Zeno equilibria for Lagrangian hybrid systems. We note that if $z = \{(\theta^*,0)^T\}$ is an isolated Zeno equilibrium of a Lagrangian hybrid system $\mathscr{H}_{\mathbf{L}}$, then we can set $\theta^* = 0$ without loss of generality by translating coordinates.

Theorem 2: *Let $\mathscr{H}_{\mathbf{L}}$ be a Lagrangian hybrid system. If $z = \{0\}$ is an isolated Zeno equilibrium of $\mathscr{H}_{\mathbf{L}}$, then there is a neighborhood W of 0 such that for all $x \in W$ there exists a unique Zeno execution $\chi = (\Lambda, I, \rho, C)$ with $c_0(\tau_0) = x$ converging to $\{0\}$ if and only if*

1) *The coefficient of restitution $e < 1$*
2) $\operatorname{sign}(h'(0)) = \operatorname{sign}(U'(0))$

Before proving this theorem, we note that the following functions:

$$V(\theta,\dot{\theta}) = \frac{1}{2}M(\theta)\dot{\theta}^2 + U(\theta) - U(0) \tag{9}$$

$$B(\theta,\dot{\theta}) = \frac{M(0)}{U'(0)}\dot{\theta} + \sqrt{\frac{M(0)^2}{U'(0)^2}\dot{\theta}^2 + 2\frac{M(0)}{U'(0)}\theta} \tag{10}$$

and constants:

$$a = \frac{1}{2} \tag{11}$$

$$b = 2\sqrt{2}\left|\frac{\sqrt{M(0)}}{U'(0)}\right| \tag{12}$$

$$r = e^2 \tag{13}$$

will be shown to satisfy the conditions in Theorem 1, thus giving a proof of Zeno behavior in a neighborhood Z of 0.

Proof:

($\Leftarrow$) We show that when $\operatorname{sign}(h'(0)) = \operatorname{sign}(U'(0))$ and $e < 1$, the functions and constants as by equations (9)-(13), satisfy all the conditions of Theorem 1.

First, since $f_L(0) \neq 0$ and $\{0\}$ is an isolated Zeno equilibrium, we can choose an open ball W around 0 such that W contains no other Zeno equilibria and W contains no equilibria of f_L. Let W_1 be the projection of W on the θ-axis.

We will also repeatedly use the fact that elements in $(G_h \cup R_h(G_h)) \cap W$ take the form $(0,\dot{\theta}) \in \mathbb{R}^2$. Indeed, if $(\theta,\dot{\theta})^T \in G_h \cap W$, then $h(\theta) = 0$, by definition of G_h. By **L**3, $h(\theta) = 0$ imples that $(\theta,0)^T$ is a Zeno equilibrium in the ball W. Therefore $\theta = 0$ since, 0 is the only Zeno equilibrium in W. Elements of $R_h(G_h) \cap W$ have the same form by **L**3.

EC1: Proving that V is positive on $W \setminus \{0\}$ is equivalent to showing that $U(\theta) > U(0)$ on $W_1 \setminus \{0\}$. By taking the Taylor series expansion of $h(\theta)$ about $\theta = 0$, it follows that $\operatorname{sign}(h(\theta)) = \operatorname{sign}(h'(0)\theta)$. Now, from the Taylor series expansion of U, we see that

$$U(\theta) - U(0) = U'(0)\theta + O(\theta^2). \tag{14}$$

Since we are assuming that $\operatorname{sign}(h'(0)) = \operatorname{sign}(U'(0))$, $h(\theta) > 0$ implies $h'(0)\theta > 0$ and thus $U'(0)\theta > 0$ on $W_1 \setminus \{0\}$. So $U(\theta) - U(0) > 0$ on $W_1 \setminus \{0\}$ and **EC**1 holds.

EC2: Follows from conservation of energy.

EC3: Since Θ is one dimensional by **L**1, the vector field f_L becomes:

$$\begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = f_L(\theta,\dot{\theta}) = \begin{pmatrix} \dot{\theta} \\ -\frac{1}{M(\theta)}\left(\frac{1}{2}M'(\theta)\dot{\theta}^2 + U'(\theta)\right) \end{pmatrix}. \tag{15}$$

Since this vector field has no equilibrium points:

$$\ddot{\theta}(0,0) = -\frac{U'(0)}{M(0)} \neq 0. \tag{16}$$

Moreover, $M(0) > 0$ and $U'(0)\theta \geq 0$ imply $\frac{M(0)}{U'(0)}\theta \geq 0$.

So for $(\theta, \dot{\theta})^T \in W$, we have

$$\sqrt{\frac{M(0)^2}{U'(0)^2}\dot{\theta}^2 + 2\frac{M(0)}{U'(0)}\theta} \geq \left|\frac{M(0)}{U'(0)}\dot{\theta}\right|.$$

Therefore, defining B as in equation (10), we have $B(\theta, \dot{\theta}) \geq 0$ on W. Thus we have shown that **EC**3 holds.

EC4: Taking the time derivative of B, we see that

$$\dot{B}(\theta, \dot{\theta}) = \frac{M(0)}{U'(0)}\ddot{\theta} + \frac{\frac{M(0)}{U'(0)}\dot{\theta}}{\sqrt{\frac{M(0)^2}{U'(0)^2}\dot{\theta}^2 + 2\frac{M(0)}{U'(0)}\theta}}\left(1 + \frac{M(0)}{U'(0)}\ddot{\theta}\right)$$

From equation (16), it follows that

$$\frac{M(0)}{U'(0)}\ddot{\theta}(0,0) = -1.$$

Therefore $\dot{B}$ is continuous at $(0,0)^T$ and $\dot{B}(0,0) = -1$. Thus $\dot{B}(\theta, \dot{\theta}) < 0$ in W and **EC**4 holds.

ED1 and **C**1: $V(R_h(0, \dot{\theta})) = e^2 V(0, \dot{\theta})$, by **L**3 of Proposition 1, and the definition of V.

C2: For $(0, \dot{\theta})^T \in R_h(G_h) \cap W$,

$$\begin{aligned} B(0,\dot{\theta}) &= \frac{M(0)}{U'(0)}\dot{\theta} + \sqrt{\left(\frac{M(0)}{U'(0)}\dot{\theta}\right)^2} \\ &\leq 2\left|\frac{M(0)}{U'(0)}\right||\dot{\theta}| \\ &= 2\sqrt{2}\left|\frac{\sqrt{M(0)}}{U'(0)}\right| V(0,\dot{\theta})^{1/2}. \end{aligned}$$

Thus the relative size condition **C**2 holds with constants defined by equations (11), (12).

($\Rightarrow$) Suppose that there is a neighborhood W of 0 such that for all $x \in W$ there exists a unique Zeno execution $\chi = (\Lambda, I, \rho, C)$ with $c_0(\tau_0) = x$ that converges to $\{0\}$.

$\mathrm{sign}(h'(0)) = \mathrm{sign}(U'(0))$: Assume that $\mathrm{sign}(h'(0)) \neq \mathrm{sign}(U'(0))$. Since $\{0\}$ is a Zeno equilibrium, $U'(0) \neq 0$ by (16). Similar to the proof of **EC**1, $h(\theta) > 0$ implies $h'(0)\theta > 0$ and $U'(0)\theta < 0$ near 0. So given any neighborhood W of 0, there exists a neighborhood N of 0 such that $N \times \{0\} \subset W$ and for all $\theta \in N \setminus \{0\}$, $U(\theta) < U(0)$. If χ satisfies $c_0(\tau_0) = (\theta_0, 0)^T \in N \times \{0\}$, then $c_i(t)$ never reaches the open set

$$\{(\theta, \dot{\theta})^T : V(\theta, \dot{\theta}) > U(\theta_0)\}$$

which contains the origin. Thus χ does not converge to the origin.

$e < 1$: Now assume that $\mathrm{sign}(h'(0)) = \mathrm{sign}(U'(0))$ but $e = 1$. Then there is a neighborhood W of the origin such that $V(\theta, \dot{\theta}) > 0$ on $W \setminus \{0\}$ from the proof of **EC**1. If an execution χ has $c_0(\tau_0) = (\theta_0, \dot{\theta}_0) \in W \setminus \{0\}$, then $V(c_i(t)) = V(\theta_0, \dot{\theta}_0)$ for all $i \in \Lambda$ and $t \in I_i$. Thus $c_i(t)$ never reaches the open set

$$\{(\theta, \dot{\theta})^T : V(\theta, \dot{\theta}) < V(\theta_0, \dot{\theta})\}$$

which contains the origin. Thus χ does not converge to the origin. ■

Example 2: Consider the hybrid system $\mathscr{H}_{\mathrm{pend}}$, defined in Example 1, modeling a pendulum impacting the ground. From the pendulum's Lagrangian, if follows that

$$U_{\mathrm{pend}}(\theta) = mgl\sin(\theta) = mgl(h_{\mathrm{pend}}(\theta)),$$

and so

$$\mathrm{sign}(U'_{\mathrm{pend}}(0)) = \mathrm{sign}(h'_{\mathrm{pend}}(0)).$$

Therefore, the conditions of Theorem 2 hold if and only if the coefficient of restitution $e < 1$. Thus, there is a neighborhood W of $(0,0)^T$ such that for all $(\theta, \dot{\theta})^T \in W$, there is a Zeno execution $\chi = (\Lambda, I, \rho, C)$ with $c_0(\tau_0) = (\theta, \dot{\theta})$ converging to $\{(0,0)^T\}$ if and only if $e < 1$.

An analogous statement holds for the Zeno equilibrium $\{(\pi, 0)^T\}$. As a result of Theorem 2, to prove that the pendulum is Zeno we only needed to evaluate two known functions at a single point.

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