Rank Properties of Poincaré Maps for Hybrid Systems with Applications to Bipedal Walking

Eric D.B. Wendel
ericdbw@tamu.edu

Aaron D. Ames
aames@tamu.edu

Department of Mechanical Engineering
Texas A&M University
College Station, TX USA

ABSTRACT

The equivalence of the stability of periodic orbits with the stability of fixed points of a Poincaré map is a well-known fact for smooth dynamical systems. In particular, the eigenvalues of the linearization of a Poincaré map can be used to determine the stability of periodic orbits. The main objective of this paper is to study the properties of Poincaré maps for hybrid systems as they relate to the stability of hybrid periodic orbits. The main result is that the properties of Poincaré maps for hybrid systems are fundamentally different from those for smooth systems, especially with respect to the linearization of the Poincaré map and its eigenvalues. In particular, the linearization of any Poincaré map for a smooth dynamical system will have one trivial eigenvalue equal to 1 that does not affect the stability of the orbit. For hybrid systems, the trivial eigenvalues are equal to 0 and the number of trivial eigenvalues is bounded above by dimensionality differences between the different discrete domains of the hybrid system and the rank of the reset maps. Specifically, if n is the minimum dimension of the domains of the hybrid system, then the Poincaré map on a domain of dimension $m \geq n$ results in at least $m - n + 1$ trivial 0 eigenvalues, with the remaining eigenvalues determining the stability of the hybrid periodic orbit. These results will be demonstrated on a nontrivial multi-domain hybrid system: a planar bipedal robot with knees.

Categories and Subject Descriptors

G.1.0 [**Numerical Analysis**]: General—*Stability (and instability)*; I.6.8 [**Simulation and Modeling**]: Types of Simulation—*Continuous, Discrete event*

General Terms

Theory

Keywords

Stability, Periodic orbits, Poincaré maps, Robotic bipedal walking, Hybrid systems

1. INTRODUCTION

A hybrid system consists of both smooth and discrete components, and it is the interaction between these components that results in phenomena which cannot occur for smooth dynamical systems. This implies that hybrid systems are fundamentally different objects than smooth dynamical systems; for example, results on the existence and uniqueness of solutions to hybrid systems are not the same as for smooth systems [17, 18] and hybrid systems display unique behavior such as Zenoness [14]. Similarly, we may not treat the stability of hybrid system equilibria in the same way as the stability of equilibria of smooth dynamical systems; see [6] for a survey of results on the stability of hybrid systems. It is therefore natural to ask, *how does the stability and analysis of periodic orbits of hybrid systems, and particularly the associated Poincaré map, differ from that of periodic orbits of smooth dynamical systems?*

The stability of periodic orbits of smooth dynamical systems is established using two facts. First, the stability of a periodic orbit is equivalent to the stability of the discrete dynamical system defined by a Poincaré map associated with that orbit. Second, the linearization of any stable Poincaré map associated with a stable periodic orbit will always have one eigenvalue equal to 1 with the remaining eigenvalues of magnitude less than 1. The eigenvalue equal to 1 is trivial since it does not relate to the stability of the system. Intuitively speaking, the trivial eigenvalue corresponds to perturbations in the direction of the closed orbit [10, 23, 24].

The primary objective of this paper is to study properties of Poincaré maps associated with hybrid periodic orbits. The main conclusion of this work is that Poincaré maps associated with hybrid periodic orbits have fundamentally different properties from those associated with periodic orbits of smooth systems. To demonstrate this, the main result of this paper is an upper bound on the number of trivial eigenvalues of the linearization of the Poincaré map. In particular, for a hybrid system, the linearization of the Poincaré map has at least one trivial eigenvalue and all the trivial eigenvalues are equal to 0; more precisely, if n is the smallest dimension of the domains of the hybrid system, then considering the Poincaré map on a domain of dimension $m \geq n$ results in at least $m - n + 1$ trivial 0 eigenvalues. The remaining eigenvalues of the linearization of the Poincaré map of the hybrid system determine the stability of the hybrid periodic orbit. In addition, we show through a counterexample that a strict equality on the number trivial eigenvalues cannot be obtained in general and that single-domain hybrid systems have exactly one trivial eigenvalue equal to 0.

It is worth noting that although many references discuss the application of the Poincaré map to multi-domain [27] and single-domain hybrid systems [5, 7], the authors have not yet found existing work that explicitly compares the properties of Poincaré maps for hybrid systems with those of Poincaré maps for smooth systems. Therefore the main result can be considered novel and complementary to prior work, for example to [19] or [21].

Furthermore, although our definition of the Poincaré map is consistent with prior work [27], our approach is unique in that we compute the linearization of the Poincaré map by simply taking its derivative, as one might expect. This is at odds with the approach taken in the literature, since we do not require an external formula to compute the derivative of the Poincaré map. In [11, 12] and in [16] different formulas are derived under special assumptions and used to define the linearization of the Poincaré map. These formulas cannot be used here since they do not apply to multi-domain hybrid systems, especially when one of the domains has a different dimension, like the planar kneed biped discussed in Section 5.2. In particular, the linearization of the Poincaré map should be given by its derivative alone.

We begin our discussion in Section 2, where we review results on smooth dynamical systems that will be useful for proving the main result of this work. Standard references for smooth systems include [10] and [24]; a manual for computing results related to standard dynamical systems theory may be found in [23]. The results in Section 2 apply only to the smooth components of a periodic hybrid system. In Section 3 we provide basic definitions of hybrid systems and their solutions and describe the smooth and discrete components. Although alternative definitions of hybrid systems may be found in [12] or in [7], the theoretical framework we develop in Section 3 is amenable to the study of periodic hybrid systems with more than one domain. In Section 4 we consider Poincaré maps associated with periodic orbits of hybrid systems and derive general properties of their linearization with a special focus on the number of trivial eigenvalues. Finally, we illustrate our results on two examples in the final Section 5, one of which is the nontrivial two-domain planar kneed bipedal walker.

2. ORBITS OF SMOOTH SYSTEMS

As will be formally discussed in Section 3, a hybrid system consists of collections of smooth and discrete components. The smooth parts are solution curves to differential equations defined by smooth vector fields. In this section, therefore, we review standard results on orbits of smooth dynamical systems which will be necessary to our analysis of Poincaré maps for hybrid systems in Section 4, beginning with the introduction of basic notation.

Flows. Let M be a manifold and TM its tangent bundle. Let $f : M \to TM$ be a Lipschitz continuous vector field such that for the canonical projection map $\pi : TM \to M$, $\pi \circ f = \mathrm{id}_M$. We will assume that $M \subset \mathbb{R}^n$, in which case we can write the vector field in coordinates as $\dot{x} = f(x)$ with $x \in M \subset \mathbb{R}^n$ where necessarily $\dot{x} \in T_x M$. A smooth function $g : M \to N$ between manifolds induces a map between the tangent space $Dg(x) : T_x M \to T_{g(x)} N$; in coordinates, this is just the Jacobian or derivative.

The unique solution to the vector field $\dot{x} = f(x)$ is a trajectory $c : I \subset [0, \infty) \to M$, which is referred to as an *integral curve* or *orbit* of $f(x)$ with initial condition $c(t_0)$ if $I = [t_0, t_1]$. The *flow* of the vector field $\dot{x} = f(x)$ is a map $\phi : I \times U \to V$, where U is some neighborhood of $x_0 = c(t_0)$, satisfying $\phi_t(x_0) = c(t)$. The flow has the following properties: for $r, s, t \in I$,

$$\begin{aligned} c(t_0) &= \phi_0(c(t_0)) \\ c(t_0 + t + s) &= \phi_{t+s}(c(t_0)) = \phi_t \circ \phi_s(c(t_0)) \\ \phi_{-r} \circ \phi_r(x_0) = x_0 &\Rightarrow \phi_{-r} = (\phi_r)^{-1} \end{aligned}$$

The flow with t a parameter, $\phi_t : U \to V$, is a diffeomorphism for all $t \in I$.

Sections. It is standard practice [24] to define the Poincaré map of a dynamical system on a certain smooth hypersurface that we construct, if possible, through a point of the flow.

Definition 1. A *local section* of a vector field $\dot{x} = f(x)$ on M is a smooth codimension-1 submanifold of M given by:

$$S = \{x \in M \mid h(x) = 0 \quad \text{and} \quad L_f h(x) \neq 0\}$$

where $h : M \to \mathbb{R}$ is a C^1 function and $L_f h$ is the Lie derivative. More generally, any submanifold $N \subset M$ is said to be *transverse* to the flow (or vector field f) if $f(x)$ is not in $T_x N$.

In fact, we may construct a local section through *any* point of a flow (that is not an equilibrium point) as a result of the following lemma from [25].

LEMMA 1 (SMALE, 1963). *For $f(x)$ a vector field defined on a smooth manifold M, if for some $x \in M$, $f(x) \neq 0$, then there exists a local section S through x, i.e., $x \in S$ with S a local section.*

Time-to-impacts map. The next fact that will be needed is that the time for a flow to reach a local section is given by a well-defined map. We apply the following lemma from [10], the proof of which follows from a direct application of the implicit function theorem.

LEMMA 2 (HIRSCH & SMALE, 1974). *Let S be a local section and $x_0 \in M$ such that $x_1 = \phi_t(x_0) \in S$. Then there exists a unique, C^1 function $\tau : U_0 \to \mathbb{R}$ called the time-to-impact map such that for U_0 a sufficiently small neighborhood of x_0, $\phi_{\tau(x)}(x) \in S$ for all $x \in U_0$.*

The proof of this lemma [10] yields the derivative of the time-to-impact map, given by:

$$D\tau(x_0) = -\frac{Dh(x_1) D_x \phi_t(x_0)}{L_f h(x_1)}. \tag{1}$$

This will be useful throughout the course of the paper.

Remark 1. Consider the map $\phi_\tau(\,-\,) := \phi_{\tau(\,-\,)}(\,-\,)$ which by Lemma 2 takes any point in a sufficiently small neighborhood U_0 of x_0 to the local section S; that is $\phi_\tau(\,-\,) : U_0 \to$

$\phi_\tau(U_0) \subset S$. As was noted in [25], for $\phi_t(x_0)$ a closed (or periodic) orbit of a smooth dynamical system with $x_0 \in U_0 \cap S$, $\phi_\tau : U_0 \cap S \to \phi_\tau(U_0) \cap S$ is a diffeomorphism, called the *associated diffeomorphism* to S. We will prove that under certain conditions ϕ_τ is a diffeomorphism even if $\phi_t(x_0)$ is not a closed orbit. In order to do so it is necessary to consider the variational equation of the flow.

Variational Equations. The *variational equation* is the linearization of $\dot{x} = f(x)$ about a trajectory $x(t)$ with initial condition $x(t_0) = x_0$; see [23, 24] for more on the variational equation. It is a nonautonomous linear equation

$$\dot{z} = A(t)z := Df\big(x(t)\big)z \tag{2}$$

with solution $z(t) = \dot{\phi}_t(x_0)$ and fundamental matrix solution

$$\Phi_t(x_0) := D_x\phi_t(x_0) = \frac{\partial \phi_t(x_0)}{\partial x_0}$$

which satisfies

$$\dot{\phi}_t(x_0) = \Phi_t(x_0)\Phi_0^{-1}(x_0)\dot{\phi}_0(x_0) = \Phi_t(x_0)\dot{\phi}_0(x_0).$$

That is, for $x_1 = \phi_t(x_0)$,

$$\begin{aligned} \dot{\phi}_t(x_0) &= D_x\phi_t(x_0)\dot{\phi}_0(x_0), \\ f(x_1) &= D_x\phi_t(x_0)f(x_0). \end{aligned} \tag{3}$$

The fundamental matrix solution is also called the *space derivative* of the flow and is always nonsingular. The *total derivative* of the flow at $x = \phi_\tau(x_0)$ is given by

$$D\phi_\tau(x_0) = D_x\phi_t(x_0) + \dot{\phi}_\tau(x_0)D\tau(x_0) \tag{4}$$

where $D\phi_\tau(x_0) : T_{x_0}U_0 \to T_x\phi_\tau(U_0)$. The rank of this map at x_0 will be of special interest. In order to compute its rank we first note the following standard result, which may be easily proven using [24, 10].

PROPOSITION 1. *The following are equivalent:*

(C1) ϕ_t *is a closed orbit with period* T.

(C2) $D\tau(x_0) \equiv 0$.

(C3) $\Phi_T(x_0)f(x_0) = f(x_0)$.

Remark 2. It is important to note that we may not assume that Proposition 1 applies to hybrid systems. For example, the formula derived in [12] is used in [11] to recover property (C3) for the planar compass biped. However, no proof of this fact is provided and, as mentioned earlier, this formula does apply to more general hybrid systems. Indeed, one can easily use the following theorem to show that this Proposition cannot hold for hybrid systems.

Rank of the total derivative. The following theorem will be of use in Section 4 and may be used to show that multi-domain hybrid systems—under suitable assumptions and conditions—do not satisfy conditions (C1)-(C3).

THEOREM 1. *Let* S *be a local section and* $x_0 \in U_0$ *with* S *and* U_0 *as in Lemma 2. If* $\dim(M) = n$ *then* $D\phi_\tau(x_0)$ *has rank* $n-1$ *if and only if* $x(t) = \phi_t(x_0)$ *is not a closed orbit.*

PROOF. ($\Rightarrow$) Suppose that $D\phi_\tau(x_0)$ has rank $n-1$. We will show that this implies that $x(t)$ is not a closed orbit. Assume that the trajectory $x(t)$ is a closed orbit with period $T = \tau(x_0)$ and initial condition $x_0 \in U_0$. Since $x(t)$ is a closed orbit, $D\tau(x_0) = 0$ by Proposition 1, and thus equation (4) simplifies to $D\phi_\tau(x_0) = \Phi_T(x_0)$, which is a rank n matrix and nonsingular. But since $D\phi_\tau(x_0)$ has rank $n-1$ our assumption that $x(t)$ is a closed orbit must be incorrect, implying that $x(t)$ is not a closed orbit.

($\Leftarrow$) Suppose that the trajectory $x(t) = \phi_t(x_0)$ is not a closed orbit. Let $\tau(x_0) = T$ and $x_1 = \phi_T(x_0)$. Note that $x_0 \in U_0$ but not necessarily in S. By Proposition 1, it follows that $\Phi_T(x_0)f(x_0) = f(x_1)$, $f(x_1) \neq f(x_0)$ and $D\tau(x_0) \neq 0$. Finally, it is possible to choose coordinates so that $\dot{\phi}_T(x_0) = f(x_1) = (0, \ldots, 1)^T$. Taking the total derivative (4) in these coordinates, coupled with (1) yields:

$$\begin{aligned} D\phi_\tau(x_0) &= \Phi_T(x_0) + \dot{\phi}_T(x_0)D\tau(x_1) \\ &= \left(I_{n\times n} - \frac{\dot{\phi}_T(x_0)Dh(x_1)}{L_f h(x_1)}\right)\Phi_T(x_0) \\ &= \left(I_{n\times n} - \begin{bmatrix} 0_{n-1\times n-1} & 0_{n-1\times 1} \\ \mathrm{proj}(Dh(x_1)) & 1 \end{bmatrix}\right)\Phi_T(x_0) \\ &= \begin{bmatrix} I_{n-1\times n-1} & 0_{n-1\times 1} \\ -\mathrm{proj}(Dh(x_1)) & 0 \end{bmatrix}\Phi_T(x_0) \end{aligned}$$

where

$$\mathrm{proj}(Dh(x_1)) := \frac{1}{Dh(x_1)_n}\begin{pmatrix} Dh(x_1)_1 & \cdots & Dh(x_1)_{n-1} \end{pmatrix}$$

Because $\Phi_T(x_0)$ is nonsingular, it follows that:

$$\begin{aligned} \mathrm{rank}(D\phi_\tau(x_0)) &= \mathrm{rank}\left(\begin{bmatrix} I_{n-1\times n-1} & 0_{n-1\times 1} \\ -\mathrm{proj}(Dh(x_1)) & 0 \end{bmatrix}\right) \\ &= n-1 \end{aligned}$$

Moreover, for the choice of coordinates it is easy to see that $\ker(D\phi_\tau(x_0)) = \mathrm{span}(f(x_0))$. $\square$

Diffeomorphisms between sections. Now that we know the rank of $D\phi_\tau$ we may use the inverse function theorem to show that ϕ_τ restricted to a section is a diffeomorphism onto its image.

Recall that the inverse function theorem states [4] that for a function between open sets $f : W \to Z$ if, for some $w \in W$, $Df(w)$ is nonsingular then $\exists$ some $W' \subseteq W$ for which $f : W' \to f(W')$ is a diffeomorphism.

COROLLARY 1. *Let* $x_0 \in M$ *with* S_0 *a local section through* x_0, S *a local section through* $x_1 = \phi_t(x_0)$ *and* U_0 *a neighborhood of* x_0 *such that* $\phi_\tau(x) \in S$ *for all* $x \in U_0$. *Then for* $V_0 := U_0 \cap S_0$ *and* $V := \phi_\tau(U_0) \cap S$, *the restriction map* $\phi_\tau : V_0 \to V$ *is a diffeomorphism.*

PROOF. Let $\psi_0 : V_0 \to \mathbb{R}^{n-1}$ and $\psi : V \to \mathbb{R}^{n-1}$ be local coordinate maps. Note that ϕ_τ has inverse $\phi_{-\tau}$ and is bijective. The expression for $D\phi_\tau : T_{x_0}V_0 \to T_xV$ in coordinates is

$$D\phi_\tau(x_0) = D\psi \circ D\phi_\tau(x_0) \circ D\psi_0^{-1}$$

Since ψ and ψ_0 are coordinate transforms, $D\psi$ and $D\psi_0^{-1}$ both have rank $n-1$, and $\mathrm{rank}(D\phi_\tau) = n-1$ by the preceding theorem. This implies that $\mathrm{rank}(D\phi_\tau) = n-1$ and is thus invertible on all points of V_0. Since it is invertible, by the inverse function theorem ϕ_τ is a diffeomorphism from all of V_0 to V. $\square$

3. HYBRID SYSTEMS AND PERIODIC ORBITS

The goal of this paper is to study the properties of Poincaré maps for hybrid systems. Since these maps are associated with periodic orbits for hybrid systems, *i.e.*, hybrid periodic orbits, we will restrict our attention to hybrid systems with an underlying graph that is "periodic", or in the language of graphs, cyclic. We begin by introducing the notions of hybrid systems on a cycle and their executions. We will revisit the results of the previous section in this context and conclude with assumptions necessary to the following sections.

Hybrid systems and executions.

Definition 2. A *k-domain hybrid system on a cycle* is a tuple

$$\mathcal{H} = (\Gamma, D, G, R, F)$$

- $\Gamma = (Q, E)$ is a *directed cycle* such that $Q = \{q_1, \dots, q_k\}$ is a set of k vertices and $E = \{e_1 = (q_1, q_2), e_2 = (q_2, q_3), \dots, e_k = (q_k, q_1)\} \subset Q \times Q$. With the set E we define maps $\mathrm{sor} : E \to V$ which returns the source of an edge, the first element in the edge tuple, and $\mathrm{tar} : E \to V$, which returns the target of an edge or the second element in the edge tuple.
- $D = \{D_q\}_{q \in Q}$ is a collection of smooth manifolds called *domains*, where D_q is assumed to be embedded submanifolds of $\mathbb{R}^{n_q}$ for some $n_q \geq 1$.
- $G = \{G_e\}_{e \in E}$ is a collection of *guards*, where G_e is assumed to be an embedded submanifold of $D_{\mathrm{sor}(e)}$.
- $R = \{R_e\}$ is a collection of *reset maps* which are smooth maps $R_e : G_e \to D_{\mathrm{tar}(e)}$.
- $F = \{f_q\}_{q \in Q}$ is a collection of Lipschitz vector fields on D_q, such that $\dot{x} = f_q(x)$.

We will assume that the hybrid systems under consideration are *deterministic* and *non-blocking*; in the case of hybrid systems on a cycle, these assumptions tend to hold in general under reasonable assumptions [15]. This assumption allows us to treat any point of a hybrid execution as an initial condition, and to assume that executions are permitted to either reach a guard or evolve for infinite time on a domain. Since these notions are not central to questions of periodic stability, we refer the interested reader to [17] for more on these topics.

Definition 3. An *execution* of $\mathcal{H}$ is a tuple

$$\chi = (\Lambda, I, \rho, C)$$

- $\Lambda = \{0, 1, 2, 3, \dots\} \subseteq \mathbb{N}$ is a finite or infinite indexing set.
- $I = \{I_i\}_{i \in \Lambda}$ such that with $|\Lambda| = N$, $I_i = [t_i, t_{i+1}] \subset \mathbb{R}$ and $t_i \leq t_{i+1}$ for $0 \leq i < N - 1$. If N is finite then $I_{N-1} = [t_{N-1}, t_N]$ or $[t_{N-1}, t_N)$ or $[t_{N-1}, \infty)$, with $t_{N-1} \leq t_N$.
- $\rho : \Lambda \to Q$ is a map such that $e_{\rho(i)} := (\rho(i), \rho(i+1)) \in E$.
- $C = \{c_i\}_{i \in \Lambda}$ is a set of continuous trajectories where each c_i is the integral curve of the vector field $f_{\rho(i)}$ on $D_{\rho(i)}$.

It is required that $\forall\, i, i+1 < |\Lambda| - 1$,

- $c_i(t_{i+1}) \in G_{(\rho(i), \rho(i+1))} = G_{e_{\rho(i)}}$
- $R_{e_{\rho(i)}}(c_i(t_{i+1})) = c_{i+1}(t_{i+1})$

and $\forall\, t \in I_i$ and $i \leq |\Lambda| - 1$, $c_i(t) \in D_{\rho(i)}$.

Note that the discrete component of the initial condition of the execution is $\rho(0) = q_i$ for $i \in \{1, \dots, k\}$ and the continuous component is the trajectory with initial condition $c_0(t_0) \in D_{q_i}$.

Hybrid periodic orbits. The orbits of hybrid systems are fundamentally different than those for smooth dynamical systems. In particular, since the flow on a domain is mapped to the next as soon as it reaches the guard, we may not assume that a hybrid periodic orbit is closed under any flow. This is in contrast with periodic orbits of smooth systems which are by definition closed under the flow. Clearly, Proposition 1 does not include hybrid systems.

Definition 4. A *hybrid periodic orbit* $\mathcal{O} = (\Lambda, I, \rho, C)$ with period T is an execution of the k-domain hybrid system on a cycle $\mathcal{H}$ such that for all $n \in \Lambda$,

- $\rho(n) = \rho(n + k)$,
- $I_n + T = I_{n+k}$,
- $c_n(t) = c_{n+k}(t + T)$.

Application of results from Section 2. The results of Section 2, and specifically Lemma 1, Lemma 2 and Corollary 1, apply to the integral curves of a hybrid execution, *i.e.*, its "continuous component", allowing us to draw the following two conclusions.

(H1) Given any execution χ of $\mathcal{H}$, for every $c_n \in C$ we may construct a local section S_0^n through $c_n(t_n)$ and a section $S^n \subset G_{e_{\rho(n)}}$ through $c_n(t_{n+1})$.

We may of course define sections through any point of a hybrid execution, but we are particularly interested in the sections through the endpoints of the flow on each domain.

(H2) Let $\phi_t^{\rho(n)}$ denote the flow of the autonomous vector field $f_{\rho(n)} \in F$ on $D_{\rho(n)}$. Since

$$c_n(t_{n+1}) = \phi_{t_{n+1} - t_n}^{\rho(n)}(c(t_n)) \in S^n,$$

there exists a neighborhood U_0^n about $c_n(t_n)$ and a time-to-impact map $\tau^n : U_0^n \to \mathbb{R}$ such that $\phi_{\tau^n}^{\rho(n)}$ maps $V_0^n := U_0^n \cap S_0^n$ diffeomorphically to $V^n := \phi_{\tau^n}^{\rho(n)}(U_0^n) \cap S^n$, where we write $\phi_\tau^{\rho(n)}(\,-\,) := \phi_{\tau^n(\,-\,)}^{\rho(n)}(\,-\,)$ for ease of notation.

Assumptions. Since we are interested in studying Poincaré maps, we begin by assuming that the system has a hybrid periodic orbit.

(A1) $\mathcal{H}$ has a hybrid periodic orbit $\mathcal{O} = (\Lambda, I, \rho, C)$.

The following two assumptions ensure that the guards and reset maps are sufficiently "well-behaved." That is, we assume that for every $e \in E$,

(A2) G_e is a section, *i.e.*, G_e is transverse to the vector field $f_{\mathrm{sor}(e)}$ and $\dim(G_e) = \dim(D_{\mathrm{sor}(e)}) - 1$.

(A3) R_e has constant rank r_e and $R_e(G_e)$ is a submanifold of $D_{\mathrm{tar}(e)}$.

Note that since R_e has constant rank, and because $R_e : G_e \to R(G_e)$ is obviously surjective, R_e is a submersion onto its image. This implies that:

$$r_e = \mathrm{rank}(R_e) = \dim(R(G_e)) \leq \dim(D_{\mathrm{sor}(e)}) - 1.$$

with the last inequality following from **(A2)** since the dimension of $R_e(G_e)$ is at most the dimension of G_e.

Implications of assumptions. The following implications of the results in Section 2 will be important in proving the main results of this paper.

Under assumption **(A1)**, and as a result of the fact that $c_n(t_n) = c_{n+k}(t_{n+k})$ and $c_n(t_{n+1}) = c_{n+k}(t_{n+k+1})$, the elements S_0^n, S^n, U_0^n, τ^n, V_0^n and V^n can be indexed by the vertex set Q of the graph Γ of $\mathcal{H}$ rather than the indexing set Λ (since, for example, one can take $S^n = S^{n+k}$). Coupling this with assumptions **(A2)** and **(A3)**, **(H1)** and **(H2)** can be restated in the following manner:

(HA1) Given any hybrid periodic orbit $\mathcal{O}$ of $\mathcal{H}$, for every $c_n \in C$ we may construct a local section $S_0^{\rho(n)}$ through the initial condition $c_n(t_n)$ and a local section $S^{\rho(n)} := G_{e_{\rho(n)}}$ through $c_n(t_{n+1}) = \phi_\tau^{\rho(n)}(c_n(t_n))$, where $\tau^{\rho(n)} : U_0^{\rho(n)} \to \mathbb{R}$ and again we adopt the notation $\phi_\tau^{\rho(n)}(\,-\,) := \phi^{\rho(n)}_{\tau^{\rho(n)}(\,-\,)}(\,-\,)$.

(HA2) There exists a neighborhood $U_0^{\rho(n)}$ about $c_n(t_n)$ such that $\phi_\tau^{\rho(n)}$ maps $V_0^{\rho(n)} := U_0^{\rho(n)} \cap S_0^{\rho(n)}$ diffeomorphically to $V^{\rho(n)} := \phi_\tau^{\rho(n)}(U_0^{\rho(n)}) \cap G_{e_{\rho(n)}}$.

4. POINCARÉ MAPS FOR HYBRID SYSTEMS

In this section we introduce Poincaré maps for hybrid systems, and study the properties of these maps. In particular, since we are interested in the stability of hybrid periodic orbits, we consider the linearization of Poincaré maps for hybrid systems. The main result of this paper is an upper bound on the number of trivial eigenvalues of the linearization based upon dimensionality differences between the different domains of the hybrid system and the minimum rank of the reset maps.

Poincaré maps for smooth systems. Recall (*cf.* [10] or [24]) that a Poincaré map for a smooth dynamical system is defined on a section S of the flow, say through some initial condition x_0. Recall from Section 2 that if the system evolves on some subset of $\mathbb{R}^n$ then $\dim(S) = n - 1$. If the flow intercepts S at least once more then we may define the *first-return map* $x_0 \mapsto \phi_\tau(x_0)$. By the proof of Lemma 2 all points of $S_0 = U_0 \cap S \neq \emptyset$ also reach S, where U_0 is some sufficiently small neighborhood of x_0. Then we define $P : S_0 \to S$, such that for all $x \in S_0$, $P(x) := \phi_\tau(x)$. The Poincaré map thus defines a discrete dynamical system $x_{k+1} = P(x_k)$.

The importance of Poincaré maps is that they allow one to determine the stability of periodic obits. In particular, if $\phi_t(x_0)$ is a closed orbit then $\phi_\tau(x_0) = x_0$ and the Poincaré map has a fixed point $x_0 = P(x_0)$. The stability of the periodic orbit is equivalent to the stability of the discrete time dynamical system $x_{k+1} = P(x_k)$ at the fixed point x_0. This, in turn, is equivalent to the stability of the linearization of this nonlinear discrete time dynamical system: $x_{k+1} = DP(x_0)x_k$ which can be determined by simply considering the eigenvalues of $DP(x_0)$. In particular, if we compute $DP(x_0)$ in natural coordinates in $\mathbb{R}^n$ then one eigenvalue of the linearization will be trivial with value 1 corresponding to perturbations out of the section in the direction of the vector field at x_0; this can be seen by (C3) in Proposition 1 since for closed orbits $f(x_0)$ is an eigenvector of the fundamental solution matrix. The other eigenvalues are nontrivial and determine stability; if they all have magnitude less than 1 the discrete time system is stable, so the nonlinear system is stable so the periodic orbit is stable. The additional advantage of considering the Poincaré map and its linearization is that it is easy and numerically robust to compute its eigenvalues.

Poincaré maps for hybrid systems. We now define Poincaré maps for hybrid systems by considering the first-return map of a hybrid periodic orbit. The following is not a novel definition of Poincaré maps and may be compared with similar definitions found in [27].

Definition 5. Let $\mathcal{O}$ be a hybrid periodic orbit of $\mathcal{H}$ with initial condition $\rho(0) = q$ and $x_0 = c_0(t_0) \in V^q \subset S^q$, where V^q is the neighborhood in S^q containing x_0 as defined in **(HA1)** and **(HA2)**. The *hybrid Poincaré map* $P_q : V^q \to S^q$ is the partial function:

$$P_q(x) = \phi_\tau^{\rho(k)} \circ R_{e_{\rho(k-1)}} \circ \cdots \circ \phi_\tau^{\rho(i+1)} \circ R_{e_{\rho(i)}} \circ \cdots \circ \phi_\tau^{\rho(1)} \circ R_{e_{\rho(0)}}(x), \quad (5)$$

where, of course, $\rho(k) = \rho(0) = q$.

Remark 3. Note that P_q is a partial function because there is no guarantee that the neighborhoods V_0^q on each domain line up with the image of the reset map from the previous domain, *i.e.*, it is not yet guaranteed that $R_e(V^{\mathrm{sor}(e)}) \subset V_0^{\mathrm{tar}(e)}$. Of course, one can ensure that this property holds by decreasing the size of $U_0^{\mathrm{sor}(e)}$ as needed, thus making P_q a function rather than a partial function. As we will see, the fact that the Poincaré map is a partial function will restrict our ability to determine its exact rank.

Remark 4. Of course, we may consider other Poincaré maps defined from arbitrary sections in D_q. It is easy to see that the linearization of any two Poincaré maps defined from arbitrary sections in the same domain D_q will share the same eigenvalues, and thus the same stability properties. This is a consequence of Corollary 1, which asserts that the flow ϕ_t defines a diffeomorphism between any two sections in a given domain. Denote the derivative of this diffeomorphism by Q. If P_q and P'_q are the Poincaré maps defined from two sections in D_q then $DP'_q = QDP_qQ^{-1}$, implying that the linearizations have the same eigenvalues.

This basic result will hold for any hybrid system $\mathcal{H}$, as long as the Poincaré maps are defined from sections in the same domain.

Stability of hybrid periodic orbits. Since P_q defines a map $P_q : V^q \to S^q$ one obtains a discrete dynamical system associated to $\mathcal{O}$ given by $x_{k_1} = P_q(x_k)$. It is important to establish, as with smooth dynamical systems, that the stability of this discrete time system directly relates to the stability of the periodic orbit $\mathcal{O}$; note that the definition of the stability of a hybrid periodic orbit is completely analogous to the definitions for smooth systems [13] and has been provided for single-domain hybrid systems in [9] and [2]. The relationship between the stability of a hybrid periodic orbit and the stability of the hybrid Poincaré map has been considered in the case when the hybrid system $\mathcal{H}$ only has one domain, *i.e.*, $Q = \{q\}$ has already been considered in [19]. In particular, there is the following result:

THEOREM 2 (MORRIS & GRIZZLE, 2005). *Let $\mathcal{H}$ be a single-domain hybrid system, i.e., $Q = \{q\}$. Then $x^* = P_q(x^*)$ is an exponentially stable fixed point of the Poincaré map $P_q : V^q \to S^q$ if and only if $\mathcal{O}$ is exponentially stable.*

Although this theorem is proven for single-domain hybrid systems, the proof relies on arguments that are not specific to single-domain hybrid systems. In particular, following from the proof of this theorem in [19], if the hybrid periodic orbit $\mathcal{O}$ is exponentially stable then necessity is proven by showing that the distance from any point on S_0^q near x^* to $\mathcal{O}$ shrinks exponentially with each iteration of the Poincaré map—this can be similarly shown for the hybrid Poincaré map defined for multi-domain systems since it is simply a local argument on the section S_0^q. Conversely, if the fixed point x^* is exponentially stable, then sufficiency follows from using the fact that the vector field on the domain is locally Lipschitz to find an appropriate bound on the distance between $\mathcal{O}$ and a neighborhood of x^* in S_0^q—for multiple domains, one would simply consider the bound obtained from the Lipschitz constants on each domain to again obtain a bound on the distance between $\mathcal{O}$ and a neighborhood of x^* in S_0^q. Therefore, Theorem 2 and its proof can be easily extended to multi-domain hybrid systems.

Poincaré maps of single-domain hybrid systems. Using Theorem 2, we can establish the stability of a hybrid periodic orbit by considering the eigenvalues of the linearization of the Poincaré map, $DP_q(x_0)$. Our main result addresses the question: *what are the trivial eigenvalues of the hybrid Poincaré map associated with this orbit and how many of them are there?* We will show that in the case of hybrid systems there is at least one trivial eigenvalue equal to 0, rather than exactly one trivial eigenvalue equal to 1 as in the case of smooth systems.

We begin by considering the case of a hybrid system $\mathcal{H}$ with one domain; these are often referred to as *simple* hybrid systems and have been well-studied in the context of hybrid mechanical systems [22].

THEOREM 3. *Let $\mathcal{H}$ be a single-domain hybrid system, i.e., $Q = \{q\}$ and $E = \{e = (q, q)\}$. If $R_e(G_e)$ is transverse to f_q then*

$$\operatorname{rank}(DP_q(x^*)) = \operatorname{rank}(R_e) \leq \dim(D_q) - 1$$

with x^* a fixed point of the hybrid Poincaré map P_q.

PROOF. Let the hybrid periodic orbit $\mathcal{O}$ have initial condition $x^* \in V^q$. Since $\mathcal{H}$ is a single-domain hybrid system, $P_q = \phi_\tau^q \circ R_e$ with $P_q : V^q \to G_e$, and $P_q(x^*) = x^*$. We know G_e is a section from assumption **(A2)**, so it follows that $\operatorname{rank}(R_e) = \dim(R_e(G_e)) \leq \dim(D_q) - 1$. If $R_e(G_e)$ is transverse to f_q then we may assume that S_0^q is chosen so that $R_e(G_e) \subset S_0^q$. Therefore, we can view R_e as a well-defined function $R_e : V^q \to R_e(G_e) \cap V_0^q$.

Now let $\phi_\tau^q|_{G_e} : R_e(G_e) \cap V_0^q \to V^q$ denote the restriction of the diffeomorphism $\phi_\tau^q : V_0^q \to V^q$. We can write the Poincaré map as

$$P_q = \phi_\tau^q \circ R_e = \phi_\tau^q|_{G_e} \circ R_e.$$

We are interested in the rank of

$$DP_q(x^*) = D\phi_\tau^q|_{G_e}(R_e(x^*))DR_e(x^*).$$

Since $\operatorname{rank}(DR_e(x^*)) = \operatorname{rank}(R_e) \leq \dim(D_q) - 1$ and $DR_e(x^*)$ is surjective (since R_e maps to a subset of $R_e(G_e)$ and has constant rank by **(A3)**), it follows that[1]

$$\operatorname{rank}(DP_q(x^*)) = \operatorname{rank}\left(D\phi_\tau^q|_{G_e}(R_e(x^*))\right).$$

Now let $\iota : R_e(G_e) \cap V_0^q \to V_0^q$ be the inclusion which implies that $\operatorname{rank}(\iota) = \dim(R_e(G_e)) = \operatorname{rank}(R_e)$. Since $\phi_\tau^q : V_0^q \to V^q$ is a diffeomorphism and $\phi_\tau^q|_{G_e} = \phi_\tau^q \circ \iota$,

$$\operatorname{rank}(\phi_\tau^q|_{G_e}) = \operatorname{rank}(\phi_\tau^q \circ \iota) = \operatorname{rank}(\iota),$$

establishing the desired result. □

If we assume that R_e is also an embedding then $\operatorname{rank}(R_e) = \dim(G_e) = \dim(D_q) - 1$ and so

$$\operatorname{rank}(DP_q(x^*)) = \dim(D_q) - 1.$$

That is, the linearization of the hybrid Poincaré map has exactly one trivial eigenvalue with value 0. It will be shown through a counterexample in Section 5.1 that a similar equality is not guaranteed for multi-domain hybrid systems.

Multi-domain hybrid systems. The goal now is to better understand the rank of the hybrid Poincaré map for multi-domain hybrid systems. We will show that the trivial eigenvalues of the linearization of the hybrid Poincaré map are 0, as in the single-domain case. However, unlike the single-domain case, it is only possible, in general, to establish upper and lower bounds on the number of trivial eigenvalues.

We begin by isolating the terms in the Poincaré map and its derivative that appear due to each edge in the directed cycle.

Definition 6. Let $e = (q, q') \in E$. The *edge map* $P_e : V^q \to V^{q'}$ is given by:

$$P_e := \phi_\tau^{q'} \circ R_e,$$

with V^q and $V^{q'}$ given as in **(HA2)**. It follows that if $q = \rho(0)$ then the hybrid Poincaré map is given by

$$P_q = P_{e_{\rho(k-1)}} \circ \cdots \circ P_{e_{\rho(1)}} \circ P_{e_{\rho(0)}}$$

We can establish both upper and lower bounds for P_e. First, it is necessary to note a basic fact from linear algebra.

[1]For an $m \times n$ matrix A and an $n \times p$ matrix B, $\operatorname{rank}(AB) = \operatorname{rank}(A)$ if B has rank n, or if B is surjective.

If A_i, $i = 1, \dots, k$ are $n_{i-1} \times n_i$ matrices then

$$\text{rank}\left(\prod_{i=1}^{k} A_i\right) \leq \min_{i\in\{1,\dots,k\}}\{\text{rank}(A_i)\} \quad (6)$$

$$\text{rank}\left(\prod_{i=1}^{k} A_i\right) \geq \sum_{i=1}^{k}\text{rank}(A_i) - \sum_{i=1}^{k} n_i \quad (7)$$

Using this, we establish the following bounds on the rank of the edge map.

Lemma 3. *If $\dim(D_q) \leq \dim(D_{q'})$ and $R_e(G_e)$ is transverse to the flow $f_{q'}$ then*

$$\text{rank}(P_e) = \text{rank}(R_e),$$

and if $\dim(D_q) > \dim(D_{q'})$ then

$$\text{rank}(P_e) \geq \text{rank}(R_e) - 1$$

Proof. Let $\dim(D_q) \leq \dim(D_{q'})$ and $R_e(G_e)$ be transverse to the flow $f_{q'}$. Then as in the proof of Theorem 3, because G_e is a section and $R_e(G_e)$ is transverse to $f_{q'}$, we may assume that $S_0^{q'}$ is chosen so that $R_e(G_e) \subset S_0^{q'}$. We can therefore view R_e as a function $R_e : V^q \to R_e(G_e) \cap V_0^{q'}$ and let $\phi_\tau^{q'}|_{G_e} : R_e(G_e) \cap V_0^{q'} \to V^{q'}$ be the restriction of the diffeomorphism $\phi_\tau^{q'} : V_0^{q'} \to V^{q'}$. Writing $P_e = \phi_\tau^{q'}|_{G_e} \circ R_e$ allows one to show that

$$\text{rank}(P_e) = \text{rank}(R_e),$$

again by the same reasoning in the proof of Theorem 3, *i.e.*, because $D\phi_\tau^{q'}|_{G_e}$ has full column rank equal to $\text{rank}(R_e)$.

Now let $\dim(D_q) > \dim(D_{q'})$. In this case, for $x \in V^q$, $D\phi_\tau^{q'}(R_e(x))$ and $DR_e(x)$ can be expressed in coordinates as $\dim(D_{q'}) - 1 \times \dim(D_{q'})$ and $\dim(D_{q'}) \times \dim(D_q) - 1$ matrices. Moreover, since $\text{rank}(\phi_\tau^{q'}) = \dim(D_{q'}) - 1$ it follows from (7) that

$$\text{rank}(P_e) \geq \text{rank}(R_e) + \text{rank}(\phi_\tau^{q'}) - \dim(D_{q'}) = \text{rank}(R_e) - 1$$

as desired. □

Remark 5. Lemma 3 states that we cannot determine the exact rank of an edge map if the dimension of the target domain is strictly less than the dimension of the source domain. Since we cannot determine the exact rank of all such edge maps, we cannot determine the exact rank of the Poincaré map.

Before stating the main theorem for multi-domain hybrid systems, we first note that in order to obtain a tighter lower bound on the rank of P_q we need to keep track of the number of edges which have target domains that are smaller in dimension than their source domains, as this negatively affects the bound on P_q, as a consequence of Lemma 3.

Definition 7. Let m be the number of *decreasing* edges for which R_e maps from a higher dimensional domain to a lower dimensional domain. Then m is given by

$$m = \left|\{e \in E : \dim(D_{\text{sor}(e)}) > \dim(D_{\text{tar}(e)})\}\right|$$

Theorem 4. *Let $\mathcal{H}$ be a hybrid system satisfying assumptions **(A1)-(A3)** with x^* a fixed point of the hybrid Poincaré map P_q. For any $q \in Q$,*

$$\text{rank}(DP_q(x^*)) \leq \min_{e\in E}\{\text{rank}(R_e)\} \leq \min_{q\in Q}\{\dim(D_q) - 1\}.$$

If for every $e \in E$ such that $\dim(D_{\text{sor}(e)}) \leq \dim(D_{\text{tar}(e)})$, $G_e(R_e)$ is transverse to the flow $f^{\text{tar}(e)}$, then

$$\text{rank}(DP_q(x^*)) \geq \sum_{e\in E}\text{rank}(R_e) - m - \sum_{q'\in Q-\{q\}}(\dim(D_{q'}) - 1)$$

where m is the number of decreasing edges.

Proof. The first inequality is a result of applying (6) together with the fact that the graph is cyclic. In particular, by the definition of the edge map (Definition 6)

$$\begin{aligned}\text{rank}(P_q) &= \text{rank}(P_{e_{\rho(k-1)}} \circ \cdots \circ P_{e_{\rho(1)}} \circ P_{e_{\rho(0)}}) \quad (8)\\ &\leq \min_{e\in E}\{\text{rank}(P_e)\}.\end{aligned}$$

Now, since $P_e = \phi_\tau^{\text{tar}(e)} \circ R_e$ and $\text{rank}(\phi_\tau^{\text{tar}(e)}) = \dim(D_{\text{tar}(e)}) - 1$ by Theorem 1,

$$\text{rank}(P_e) \leq \min\{\text{rank}(R_e), \dim(D_{\text{tar}(e)}) - 1\}.$$

Therefore,

$$\text{rank}(DP_q(x^*)) \leq \min\left\{\min_{e\in E}\{\text{rank}(R_e)\}, \min_{q\in Q}\{\dim(D_q) - 1\}\right\}$$

Finally, since $\text{rank}(R_e) \leq \dim(D_{\text{sor}(e)}) - 1$ and because the graph is a cycle,

$$\min_{e\in E}\{\text{rank}(R_e)\} \leq \min_{q\in Q}\{\dim(D_q) - 1\}$$

which yields the upper bound on $\text{rank}(DP_q(x^*))$.

The second equality follows by applying (7) coupled with Lemma 3. Note that for each map $P_e : V^{\text{sor}(e)} \to V^{\text{tar}(e)}$, the derivative $DP_e(x)$ can be expressed in coordinates as a $\dim(D_{\text{sor}(e)}) - 1 \times \dim(D_{\text{tar}(e)})$ matrix. By (8) and (7) we have[2]

$$\text{rank}(P_q) \geq \sum_{e\in E}\text{rank}(P_e) - \sum_{q'\in Q-\{q\}}(\dim(D_{q'}) - 1).$$

Now by Lemma 3 and with m the number of decreasing edges as in Definition 7, it is easy to see that the desired lower bound on $\text{rank}(P_q)$ is obtained. □

We may use the above Theorem to guarantee that a given Poincaré map will have *at least* a certain number of trivial eigenvalues equal to 0. However, guaranteeing the exact number is not possible in general. That is, unlike single-domain hybrid systems or even smooth dynamical systems, there are hybrid systems such that

$$\text{rank}(DP_q(x^*)) < \min_{e\in E}\{\text{rank}(R_e)\}.$$

The first example of the following section serves as an example of this strict inequality, and provides a counter-example to the assertion that the rank of the Poincaré map can be exactly determined for hybrid systems.

5. APPLICATIONS

In this section we apply the conclusions of the preceding sections on two separate applications. The first application is a simple example of a hybrid system whose Poincaré maps have rank strictly less than the upper bound in Theorem 4. In the second example we determine that the Poincaré maps

[2]Note that $\dim(D_q) - 1$ is not subtracted from the sum of the ranks since it corresponds to n_0 and n_k using the notation of the dimensions of the matrices given in (7).

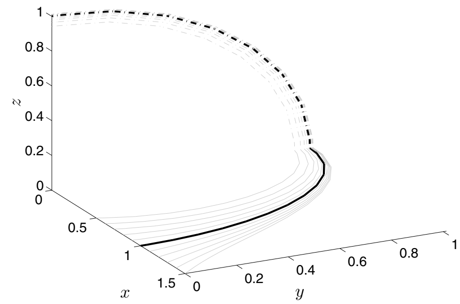

Figure 1: Limit cycle of the two-domain system in Section 5.1. All grey trajectories in D_1 (shown solid) and D_2 (dash-dotted) converge to the stable periodic orbit (black) after one traversal of the cycle.

associated with a nontrivial model of a planar bipedal robot with knees have rank equal to the upper bound established in the main theorem, and it is shown that the periodic orbit is stable.

5.1 Zero-rank Poincaré maps

In this first application we emphasize two ideas. First, the following two-domain hybrid system is an example of a system that can have hybrid Poincaré maps with rank strictly less than the upper bound derived in Theorem 4. Secondly, computations will show that the rank of both Poincaré maps is equal to 0. This means that the hybrid limit cycle is not sensitive to perturbations away from it; such systems are sometimes called *superstable*, or are said to display instantaneous convergence to a limit cycle.

We define the first domain, D_1, of this two-domain hybrid system to be the upper-right quadrant of $\mathbb{R}^2$. The vector field in this domain is

$$f_1(x,y) = \left(-y + x(1-x^2-y^2),\ x + y(1-x^2-y^2)\right)^T.$$

The flow resets to the next domain when it reaches the positive y-axis, which we define to be the guard G_{e_1}. The reset map R_{e_1} projects the y-axis into $\mathbb{R}^3$ such that

$$R_{e_1}(0,y) = (0,y,0)^T$$

clearly has rank 1.

The flow in the second domain, D_2, is the linear system

$$f_2(x,y,z)^T = (-x,-z,y)^T,$$

and is permitted to flow from the y-axis in $\mathbb{R}^3$ until it reaches the xz-plane, which defines G_{e_2}. All points on the xz-plane are mapped back to the first domain by the reset map

$$R_{e_2}(x,0,z) = (x+1,0)^T,$$

which also has rank 1.

Considering Theorem 4 and Lemma 3 together yields

$$0 \le \operatorname{rank}(P_1) \le 1, \quad 0 \le \operatorname{rank}(P_2) \le 1,$$

where P_1 is the Poincaré map defined from G_{e_1} and P_2 likewise from G_{e_2}. We find that $\operatorname{rank}(P_1) = \operatorname{rank}(P_2) = 0$, through numeric computation. Therefore, equality with the upper bound of Theorem 4 is not obtained. In addition, we may interpret this to mean that all trajectories emanating from G_{e_1} or G_{e_2} will converge to the limit cycle after at most one iteration — see Figure 1.

5.2 Planar kneed bipedal robot

The application of interest to this work is a controlled planar biped with locking knees walking on flat ground, as studied in [1]. It may be considered the augmentation of the planar compass biped described in [8, 26] with an additional domain where the stance leg is locked and the non-stance leg is unlocked at the knee.

Planar biped model. The planar biped is a two-domain hybrid system on a cycle

$$\mathcal{H} = (\Gamma, D, G, R, FG)$$

with graph structure $\Gamma = \{Q = \{u,l\}, E = \{e_u = (u,l), e_l = (l,u)\}\}$ and domains $D = \{D_u, D_l\}$. In the so-called *unlocked domain* D_u, the non-stance calf pivots about the knee and we model the biped as a 3-link planar mechanism. The dynamics evolve on the tangent bundle to the configuration space $\Theta_u := \mathbb{T}^3$, which we give coordinates $\theta_u = (\theta_s, \theta_{ns}, \theta_k)^T$ with the stance leg angle denoted by θ_s, non-stance thigh angle by θ_{ns}, and non-stance calf angle by θ_k. Each angle is measured from the vertical. Since the non-stance thigh and calf are locked together in the *locked domain* D_l the dynamics evolve on the tangent bundle to the configuration space $\Theta_l := \mathbb{T}^2$ with coordinates $\theta_l = (\theta_s, \theta_{ns})^T$. We transition from D_u to D_l when the knee locks, and from D_l to D_u when the foot strikes the ground.

Each domain and guard is defined using unilateral constraint functions $h_i : D_i \to \mathbb{R}$, for $i \in Q$. In the locked domain the end of the non-stance foot is aboveground when the function $h_l : D_l \to \mathbb{R}$

$$h_l(\theta_l) = l\left(\cos(\theta_s) - \cos(\theta_{ns})\right)$$

is positive, and strikes the ground when h_l is equal to 0. The unlocked domain is subject to the constraint $h_u : D_u \to \mathbb{R}$ such that

$$h_u(\theta_u) = \theta_k - \theta_{ns}$$

is positive when the knee is bent and equal to 0 at kneestrike, when it locks. The constraint functions define our domains such that

$$D_i = \left\{ \begin{pmatrix} \theta_i \\ \dot{\theta}_i \end{pmatrix} \in TQ_i \mid h_i(\theta_i) > 0 \right\},$$

for $i \in Q$. They also define transitions to the next domain in the cycle such that

$$G_{e_i} = \left\{ \begin{pmatrix} \theta_i \\ \dot{\theta}_i \end{pmatrix} \in TQ_i \mid h_i(\theta_i) = 0 \text{ and } \dot{h}_i(\theta_i) < 0 \right\}.$$

G_{e_u} is the set of states where the leg is locking and G_{e_l} is the set of states where the swing foot is striking the ground. Note that each G_{e_i} is a codimension-1 submanifold transversal to the vector field f_i, since $\dot{h}_i = L_{f_i} h_i < 0$, for each $i \in Q$.

The reset maps $R = \{R_{e_u}, R_{e_l}\}$ model transitions from one domain to the next. We make the standard assumption that all impacts are perfectly plastic; detailed discussions of and formulas for the impact map may be found in [9, 3, 1] and so will not be repeated due to space constraints.

We derive the equations of motion on each configuration space using the Euler-Lagrange equations. This entails finding the Lagrangian on each domain, a functional

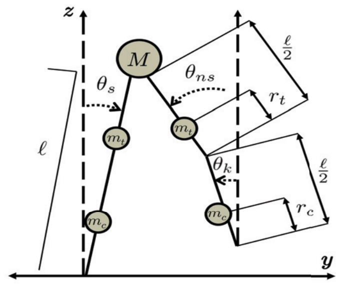

Figure 2: Biped dimensions, point-mass locations and measuring conventions.

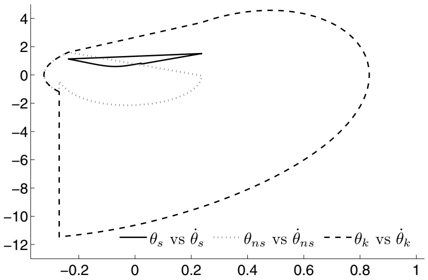

Figure 3: Biped phase portrait. In D_l the non-stance calf angle θ_k (shown dashed) is equal to the non-stance angle θ_{ns} (dashed-dotted).

$L_i : D_i \to \mathbb{R}$, such that for $i \in Q$,

$$L_i = \frac{1}{2}\dot\theta_i^T \, M_i(\theta_i)\,\dot\theta_i - V_i(\theta_i)$$

where $x_i^T = (\theta_i, \dot\theta_i)^T$. Then the Euler-Lagrange equations take the standard form [20]

$$\begin{aligned} B_i v_i &= \frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot\theta_i}\right) - \frac{\partial L_i}{\partial \theta_i} \\ &= M_i(\theta_i)\ddot\theta_i + C_i(\theta_i, \dot\theta_i)\dot\theta_i + N_i(\theta_i) \end{aligned}$$

where if $\dim(Q_i) = n$, the Coriolis matrix C_i is a $n \times n$ matrix, the gravitational torque vector N_i is $n \times 1$, and $B_i v_i$ is an appropriate control input. Since we want the biped to walk on flat ground we will use controlled symmetries [26, 1] as our control law on each domain.

The above defines $f_i(x_i)$, for every $i \in Q$ as follows:

$$\dot x_i = f_i(x_i) = \begin{pmatrix} \dot\theta_i \\ M_i^{-1}(\theta_i)\left(B_i\, v_i - N_i(\theta_i) - C_i(\theta_i, \dot\theta_i)\dot\theta_i\right) \end{pmatrix}$$

Simulation results. A periodic orbit $\mathcal{O}$ for the planar kneed biped $\mathcal{H}$ was found through exhaustive numeric search[3] after selecting a reasonable set of mass and length parameters for the biped:

$$M = m_t = 5 \text{ kg}, \qquad m_c = 0.5 \text{ kg}$$
$$L = 1 \text{ m}, \qquad r_c = 0.375 \text{ m}, \qquad r_t = 0.175 \text{ m}$$

By choosing an initial condition for this hybrid periodic orbit on the guard G_{e_u} of the unlocked domain $D_{\rho(0)} = D_u$ we ensure that our first continuous trajectory is a single point. That is,

$$\begin{aligned} c_0(t_0) = c_0(t_1) = (&0.021462,\ -0.26990,\ -0.26990, \\ &0.82882,\ -0.45645,\ -11.454)^T \end{aligned}$$

The initial condition in the locked domain after the instantaneous transition due to kneestrike is given by $R_{e_u}(c_0(t_1)) = c_1(t_1)$. The biped flows on $D_{\rho(1)} = D_l$ until footstrike, when the flow reaches the guard G_{e_l}. Theorem 1 predicts that the flow from initial condition to the guard on this domain should have rank 3, *i.e.* the total derivative $D\phi_\tau^l$ given by equation (4) should have rank 3. The eigenvalues of $D\phi_\tau^l(c_1(t_1)) = \Phi_\tau^l(c_1(t_1)) + f_l(c_1(t_2))D\tau(c_1(t_1))$ are

$$\sigma\Big(D\phi_\tau^l(c_1(t_1))\Big) = \begin{pmatrix} 1.2765 \\ -0.23479 \\ 8.8037 \times 10^{-17} \\ 0.41277 \end{pmatrix}.$$

Note that one eigenvalue is close to machine precision, meaning that it is virtually equal to 0. This allows us to conclude that the numeric results agree with Theorem 1.

At footstrike we transition back to the unlocked domain $D_{\rho(2)} = D_u$, with initial condition given by $R_{e_l}(c_1(t_2)) = c_2(t_2)$. A similar computation confirms that $D\phi_\tau^u(c_2(t_2))$ has rank 5, as expected.

Since we have already computed the total derivative on each domain, it is straightforward to compute the eigenvalues of the linearization of the Poincaré maps. The first-return map of $c_0(t_0)$ is given by

$$P_u(c_0(t_0)) = \phi_\tau^u \circ R_{e_l} \circ \phi_\tau^l \circ R_{e_u}(c_0(t_0))$$

The eigenvalues of the linearization of the Poincaré map from the guard in this domain are given by

$$\sigma\Big(DP_u(c_0(t_0))\Big) = \begin{pmatrix} 0.58775 \pm 0.50861i \\ -6.1057 \times 10^{-16} \\ 0.16871 \\ 1.1819 \times 10^{-16} \\ -6.5121 \times 10^{-18} \end{pmatrix}$$

The 3 eigenvalues close to machine precision are trivial eigenvalues, so $\text{rank}(P_u) = 3$. The remaining nontrivial eigenvalues all have magnitude less than 1, so P_u is also stable.

Similarly, the first-return map of an initial condition in G_{e_l} is given by

$$P_l(c_0(t_0)) = \phi_\tau^l \circ R_{e_u} \circ \phi_\tau^u \circ R_{e_l}(c_0(t_0)).$$

The eigenvalues of its linearization are

$$\sigma\Big(DP_l(c_0(t_0))\Big) = \begin{pmatrix} 0.58836 \pm 0.50742i \\ 0.16930 \\ -1.1201 \times 10^{-15} \end{pmatrix}$$

where $c_0(t_0) = (0.23720, -0.23720, 1.5163, 1.6023)^T$. One eigenvalue close to machine precision is trivial, implying that $\text{rank}(P_l) = 3$. The remaining eigenvalues are within the unit circle, implying that P_l is stable.

Since $\text{rank}(P_u) = \text{rank}(P_l) = 3$, equality with the upper bound derived on the rank in Theorem 4 is achieved for this particular system and the stability of each Poincaré map is determined by exactly 3 eigenvalues.

[3] An approach to finding periodic orbits of smooth systems, and to computing the space derivative Φ_τ^i in equation (3) is described in [23].

6. CONCLUSIONS AND FUTURE WORK

We have shown that the properties of Poincaré maps of hybrid systems are fundamentally different from the properties of Poincaré maps for smooth systems. The trivial eigenvalues of the linearization of a hybrid Poincaré map are equal to 0, and the number of trivial eigenvalues is bounded above by dimensionality differences between the discrete domains of the hybrid system and the minimum rank of the reset maps. We have illustrated these conclusions on a nontrivial hybrid system — a planar kneed biped — and on a simple two-domain hybrid system with rank strictly less than the derived upper bound, thereby showing that equality with the upper bound cannot be obtained for general hybrid systems, *i.e.* only upper and lower bounds on the rank can be given.

Future work will be directed towards deriving conditions that will allow us to determine the rank of the Poincaré map exactly, or at least obtain tighter upper and lower bounds on the rank. It is also important to understand how Poincaré maps defined on different sections in different domains relate to each other.

7. REFERENCES

[1] A. Ames, R. Sinnet, and E. Wendel. Three-dimensional kneed bipedal walking: A hybrid geometric approach. In *Proceedings of the 12th International Conference on Hybrid Systems: Computation and Control*, pages 16–30. Springer, 2009.

[2] A. D. Ames and R. D. Gregg. Stably extending two-dimensional bipedal walking to three dimensions. In *26th American Control Conference*, New York, NY, 2007.

[3] A. D. Ames, R. D. Gregg, E. D. Wendel, and S. Sastry. On the Geometric Reduction of Controlled Three-Dimensional Bipedal Robotic Walkers. In *Proc. 3rd IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, pages 189–202, Nagoya, Japan, July 2006.

[4] W. M. Boothby. *An Introduction to Differentiable Manifolds and Riemannian Geometry, Revised, Volume 120, Second Edition*. Academic Press, 2nd edition, Aug. 2002.

[5] B. Brogliato. *Nonsmooth mechanics: models, dynamics, and control*. Springer Verlag, 1999.

[6] G. Davrazos and N. Koussoulas. A review of stability results for switched and hybrid systems. In *Mediterranean Conference on Control and Automation*, 2001.

[7] A. F. Filippov and F. M. Arscott. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publishers, Dordrecht [Netherlands], 1988.

[8] A. Goswami, B. Thuilot, and B. Espiau. Compass-like biped robot part i: Stability and bifurcation of passive gaits. Technical Report 2996, INRIA, October 1996.

[9] J. Grizzle, G. Abba, and F. Plestan. Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. *IEEE Transactions on Automatic Control*, 46(1):51–64, 2001.

[10] M. Hirsch and S. Smale. *Differential equations, dynamical systems, and linear algebra*. Academic Press Inc, 1974.

[11] I. Hiskens. Stability of hybrid system limit cycles: Application to the compass gait biped robot. In *IEEE Conference on Decision and Control*, volume 1, pages 774–779. IEEE; 1998, 2001.

[12] I. Hiskens and M. Pai. Trajectory sensitivity analysis of hybrid systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 47(2):204–220, 2000.

[13] H. Khalil. *Nonlinear systems*. Prentice hall Englewood Cliffs, NJ, 1996.

[14] A. Lamperski and A. Ames. Lyapunov-like conditions for the existence of zeno behavior in hybrid and lagrangian hybrid systems. In *IEEE Conference on Decision and Control*, 2007.

[15] A. Lamperski and A. D. Ames. On the existence of Zeno behavior in hybrid systems with non-isolated Zeno equilibria. In *Proceeding of IEEE Conference on Decision and Control*, pages 2776–2781, 2008.

[16] R. Leine and H. Nijmeijer. *Dynamics and bifurcations of non-smooth mechanical systems*. Springer Verlag, 2004.

[17] J. Lygeros, K. Johansson, S. Sastry, and M. Egerstedt. On the existence of executions of hybrid automata. In *Decision and Control, 1999. Proceedings of the 38th IEEE Conference on*, volume 3, 1999.

[18] J. Lygeros, K. Johansson, S. Simic, J. Zhang, and S. Sastry. Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control*, 48(1):2–17, 2003.

[19] B. Morris and J. Grizzle. A restricted poincare map for determining exponentially stable periodic orbits in systems with impulse effects: Application to bipedal robots. In *44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05*, pages 4199–4206, 2005.

[20] R. Murray, Z. Li, and S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.

[21] S. Nersesov, V. Chellaboina, and W. Haddad. A generalization of poincare's theorem to hybrid and impulsive dynamical systems. In *Proceedings of the American Control Conference*, volume 2, pages 1240–1245, 2002.

[22] Y. Or and A. D. Ames. Stability of Zeno equlibria in Lagrangian hybrid systems. In *IEEE Conference on Decision and Control*, 2008.

[23] T. Parker and L. Chua. *Practical numerical algorithms for chaotic systems*. Springer-Verlag New York, Inc. New York, NY, USA, 1989.

[24] L. Perko. *Differential equations and dynamical systems*. Springer, 2001.

[25] S. Smale. Stable manifolds for differential equations and diffeomorphisms. *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.*, 17:97–116, 1963.

[26] M. Spong and F. Bullo. Controlled symmetries and passive walking. *IEEE Transactions on Automatic Control*, 50(7):1025–1031, 2005.

[27] E. R. Westervelt. *Feedback Control of Dynamic Bipedal Robot Locomotion*. Number 1 in Control and automation. CRC Press, Boca Raton, 2007.