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Data-driven control for feedback linearizable single-input systems*

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Abstract— **More than a decade ago Fliess and co-workers [1], [2], [3] proposed model-free control as a possible answer to the inherent difficulties in controlling non-linear systems. Their key insight was that by using a sufficiently high sampling rate we can use a simple linear model for control purposes thereby trivializing controller design. In this paper, we provide a variation of model-free control for which it is possible to formally prove the existence of a sufficiently high sampling rate ensuring that controllers solving output regulation and tracking problems for the approximate linear model also solve the same problems for the true and unknown nonlinear model. This is verified experimentally on the bipedal robot AMBER-3M.**

I. INTRODUCTION

The work in this paper is motivated by the control of highly dynamic cyber-physical systems such as walking robots and cars. When designing controllers for these systems it is common engineering practice to combine model-based and model-free approaches. In the automotive domain, bicycle models are typically used for control design although they fail to describe highly dynamic maneuvers involving roll, see, e.g. [4]. Similarly, when designing controllers for walking robots, it is common to use rigid body dynamic models [5], [6] that typically do not directly account for the higher order dynamics, e.g., actuator dynamics that include friction, backlash, unmodeled compliance and the role of the motor controller in the overall robot dynamics.

One advantage of combining model-based with model-free control is a natural control hierarchy that allows one to use the same model-based high level controller even when the hardware (motors, gear-boxes, etc) are changed. This approach enables one to focus on the design of high-level controllers based on idealized first principles models while neglecting several low-level considerations related to implementation. Yet, this results in a gap between the formal guarantees made at the level of the model and the actual implementation on the hardware which leverages artful and hierarchical data-driven approaches.

In this paper we take the first steps towards this model-based/model-free control hierarchy by investigating a specific model-free control technique inspired by Fliess and co-workers' work on model-free control. Starting with the papers [1], [2], [3] (see also [7]), Fliess and co-workers exploited the insight that by using sufficiently high sampling rates one can work with a nilpotent approximation of the dynamics that can easily be made linear.

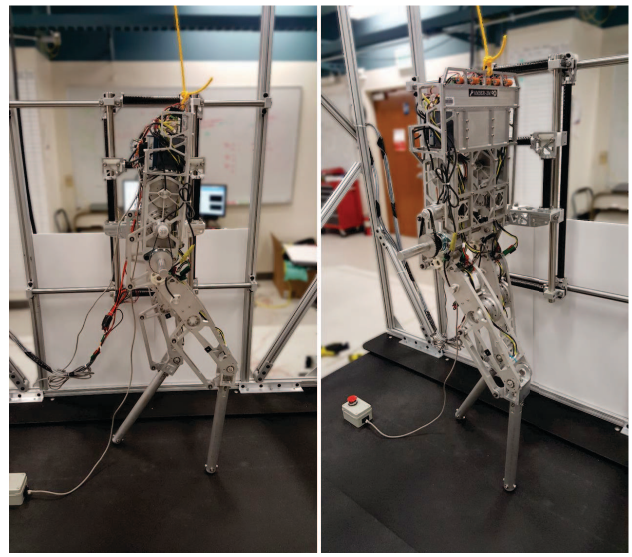

Fig. 1. AMBER-3M: the custom-built bipedal walking robot used to experimentally validate the paper's results.

With the objective of understanding the capabilities as well as the limitations of this technique we present a formulation of Fliess and co-workers model-free control along with a proof that it can be used to solve output regulation and tracking problems. The main arguments of such proof are based on the work of Nesic and co-workers [8], [9], [10] that explains how the robustness of controllers and observers designed for an approximate discrete-time model can be used to compensate for the modeling error, as long as the sampling rate is sufficiently high. As we explain in the paper, although we remain faithful to Fliess and co-workers model-free control philosophy, the control design methodology we propose has several differences: 1) we work with discrete-time controllers rather than continuous-time ones, i.e., we explicitly address how the sampling rate affects the dynamics rather than assuming it to be small enough so as to confound the continuous-time models with the sampled-data models, and 2) we do not rely on *algebraic estimation* [11], [12] since this technique is ill defined when the sampling time tends to zero and thus becomes extremely sensitive to measurement noise for small sampling times.

Rather than presenting the results in its most general form, we make several simplifying assumptions to streamline the proofs and bring out the main ideas. In addition, we work out in detail the case of systems with relative degree 2 and we experimentally validate this case by controlling a knee

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joint of AMBER-3M, a planar bipedal robot developed at AMBER Lab (see Figure 1), as detailed in Section VI.

II. PROBLEM SETUP

A. Notation

All the functions in this paper are assumed to be infinity differentiable to simplify the presentation, however the results hold under weaker differentiability assumptions. We denote the Lie derivative of the function $h : \mathbb{R}^n \to \mathbb{R}$ along the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ by $L_f h$. We also use the notation $L_f^k h$ to denote the k-th Lie derivative of h along f inductively defined by $L_f^0 h = h$ and $L_f^k h = L_f(L_f^{k-1} h)$. We denote the 2-norm of a vector $x \in \mathbb{R}^n$ by $\|x\|$.

B. Model

We consider a single-input single-output control affine nonlinear system:

$$\dot{x} = f(x) + g(x)u \quad \text{(II.1)}$$

$$y = h(x) \quad \text{(II.2)}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$. The dynamics described by f and g is unknown and we only make the following two assumptions:

1) The output $y = h(x)$ has relative degree[1] n, in other words, the system is feedback linearizable;
2) The function $L_f^n h$ is globally Lipschitz continuous and the function $L_g L_f^{n-1} h$ is constant, non-zero, and known.

Assumption 1) can be relaxed by simply requiring the output $y = h(x)$ to have well defined relative degree not necessarily equal to n. However, in such case we would require additional assumptions on the zero dynamics and a much more detailed analysis would be needed. This case and corresponding details will appear elsewhere. As we illustrate in Section VI the feedback linearizability assumption already covers cases of practical interest.

Assumption 2) can partially be relaxed. Rather than assuming $L_g L_f^{n-1} h$ to be a constant we can identify this function from data. However, this identification problem is challenging since it requires a persistently exciting input signal that may be detrimental to the stabilization problem. Moreover, $L_g L_f^{n-1} h$ is indeed constant in the practical example discussed in Section VI. The global Lipschitz continuity assumption on $L_f^n h$ cannot be substantially weakened. It was shown in [13] that if $L_f^n h$ is of the form $L_f^n h = (L_f^{n-1} h)^k$ then k must satisfy $k < n/(n-1)$ in order for stabilization, by a controller measuring y, to be possible. As n increases we see that $L_f^n h$ must essentially be a linear function of $L_f^{n-1} h$ and thus globally Lipschitz.

We will use a system with $n = 2$ as our running example. Assumption 1) results in the following model where we use the coordinates $z = (z_1, z_2) = (y, \dot{y})$:

$$\dot{z}_1 = z_2 \quad \text{(II.3)}$$

$$\dot{z}_2 = L_f^2 h(x) + L_g L_f h(x) u = a(z) + b(z)u. \quad \text{(II.4)}$$

[1]The system (II.1)-(II.2) is said to have relative degree $r \in \mathbb{N}$ if $L_g L_f^k h(x) = 0$ for all $k \leq r-2$ and $L_g L_f^{r-1} h(x) \neq 0$ for all $x \in \mathbb{R}^n$.

C. Problem formulation

The main idea introduced by Fliess and co-workers in [1], [2], [3], [7] is that we can choose a sampling rate so high that $a(z)$ and $b(z)$ can be treated as constants during a few consecutive sampling instants. Note that if $a(z)$ and $b(z)$ were indeed constant, we could explicitly integrate the model (II.3)-(II.4) to obtain:

$$z_1^a(t_k+T) = z_2^a(t_k) + T z_2^a(t_k) + \frac{1}{2}T^2(a + bu(t_k)) \quad \text{(II.5)}$$

$$z_2^a(t_k+T) = z_2^a(t_k) + T(a + bu(t_k)) \quad \text{(II.6)}$$

where $T \in [0, \tau[$ is the time elapsed since the sampling instant $t_k \in \mathbb{R}$ and before the next sampling instant $t_{k+1} = t_k + \tau$ with τ being the sampling period. The superscript a emphasizes the approximate nature of the model. Designing a stabilizing controller for this affine system is straightforward as long as we can estimate z_2 and a (recall that b is assumed to be known).

This leads to the following question answered in this paper:

Is there a sufficiently small sampling period so that a dynamic controller asymptotically stabilizing the affine model defined (II.5)–(II.6) *also asymptotically stabilizes the unknown nonlinear model* (II.1)–(II.2)*?*

III. ESTIMATION

We first address the question of estimating $L_f h$, $L_f^2 h$, ..., $L_f^r h$. For the case $n = 2$ this corresponds to estimating $z_2 = L_f h$ and $a = L_f^2 h$. We can add a as a state to the model (II.5)–(II.6) to obtain the linear model:

$$z_1^a(t_k+T) = z_2^a(t_k) + T z_2^a(t_k) + \frac{1}{2}T^2(z_3^a(t_k) + bu(t_k)) \quad \text{(III.1)}$$

$$z_2^a(t_k+T) = z_2^a(t_k) + T(z_3^a(t_k) + bu(t_k)) \quad \text{(III.2)}$$

$$z_3^a(t_k+T) = z_3^a(t_k), \quad \text{(III.3)}$$

also writen in matrix form as:

$$z^a(t_k+T) = A(T)z^a(t_k) + Bu(t_k), \quad y^a(t_k) = Cz^a(t_k).$$

Note that we regard the preceding expression as defining, not one, but a *family* of linear models parameterized by $T \in \mathbb{R}_0^+$. Once we choose a sampling time τ and fix T to be equal to τ we obtain a discrete-time model. For now, however, T is treated as a design parameter.

We can also design a *family* of Luenberger observers:

$$\widehat{z^a}(t_k+T) = A(T)\widehat{z^a}(t_k) + Bu(t_k) + L(T)(y^a(t_k) - C\widehat{z^a}(t_k)). \quad \text{(III.4)}$$

rendering the dynamics of the error $e^a = \widehat{z^a} - z^a$ asymptotically stable in the following specific sense. There exists a quadratic Lyapunov function E, a constant $\alpha_e \in \mathbb{R}^+$, and a time $\tau_e \in \mathbb{R}^+$ satisfying:

$$E(e^a(t_k+T)) - E(e^a(t_k)) \leq -\alpha_e T \|e^a\|^2, \qquad \forall T \in [0, \tau_e[. \quad \text{(III.5)}$$

Note how E provides a certificate of asymptotic stability based on an upper bound on the decrease of E that is linear in T. It is this linear dependence on T, one of the major insights introduced by Arcak and Nesic in their study of observers based on approximate models [8], that is used to build the argument for the main result in the paper.

The family of Luenberger observers (III.4) can be designed, for example, by designing a *constant* observer gain L for a continuous-time Luenberger observer based on the continuous-time model $\dot{z}_1^a = z_2^a$, $\dot{z}_2^a = z_3^a + bu$, $\dot{z}_3^a = 0$ since this guarantees the existence of a quadratic Lyapunov function E satisfying (III.5) for some $\alpha_e \in \mathbb{R}^+$ and for sufficiently small time $\tau_e \in \mathbb{R}^+$. It is not difficult to see that the same process can be applied when $n > 2$, still resulting in an inequality of the form (III.5). In Section V we will use inequality (III.5) to establish asymptotic stability for the unknown nonlinear system. Therefore, any observer resulting in an inequality of the form (III.5) can be used.

Remark 3.1: An alternative to using an observer is to use algebraic estimation [11], [12] by which we obtain the estimate $\widehat{z}$ of z via an algebraic expression of y and its iterated integrals. Although these algebraic estimators are obtained in [11], [12] via operational calculus and differential algebra, it is shown in [14] how they can be derived by resorting to linear systems theory. We summarize the discussion in [14] since it is especially relevant. Under the assumption that u is constant in between sampling times, since a and b are also assumed constant, we have (see (II.3)-(II.4)*) $\ddot{z}_2 = \frac{d}{dt}(a+bu) = 0$. Hence, $\frac{d^3}{dt^3} z_1 = 0$ and this model can be written in the classical matrix form $\dot{x} = Ax, y = Cx$ where $x = (z_1, z_2, z_3) = (z_1, \dot{z}_1, \ddot{z}_1)$ and $y = z_1$. Therefore, estimating $\dot{y} = z_2$ and estimating $a = z_3 - bu$ (we assume b and u to be known) reduces to estimating x. It then follows from classical linear systems theory (e.g. [15]) that:*

$$x(t_k+\tau) = W_r^{-1}(t_k, t_k+\tau) \int_{t_k}^{t_k+\tau} \Phi^T(s, t_k+\tau) C^T y(s) ds \tag{III.6}$$

where W_r is the reconstructability Gramian and Φ is the state transition matrix. All the formulas for algebraic estimation in [11], [12] are subsumed by (III.6) *and we now see that when the sampling time τ tends to zero, we have $\lim_{\tau \to 0} W_r(t_k, t_k+\tau) = C^T C$ which is no longer invertible and precludes the use of algebraic estimation. Moreover, as τ becomes smaller (although still nonzero), the estimate of x given by* (III.6) *becomes more sensitive to noise since W_r^{-1} becomes numerically ill conditioned.*

IV. STABILIZING THE LINEAR APPROXIMATE MODEL

We first assume that we can measure the full state of the linear model (III.1)-(III.3). It is then simple to design a family of linear controllers (parameterized by T):

$$u(t_k + T) = K(T) z^a(t_k) \tag{IV.1}$$

asymptotically stabilizing z_1^a and z_2^a to the origin and for which there exists a quadratic Lyapunov function $V(z_1^a, z_2^a)$ and a time $\tau_z \in \mathbb{R}^+$ so that the following inequality holds for all $T \in [0, \tau_z[$:

$$V(z_{12}^a(t_k + T)) - V(z_{12}^a(t_k)) \leq -\alpha_z T \|z_{12}^a(t_k)\|^2. \tag{IV.2}$$

One way of designing such controller is to use a constant matrix K for which the control law $u = Kx$ asymptotically stabilizes the continuous-time model $\ddot{z}_1^a = a + bu$. It would then follow the existence of a Lyapunov function satisfying $\dot{V} \leq -\beta \|(z_1^a, z_2^a)\|^2$. By continuity, (IV.2) will hold with, e.g., $\alpha_z = \beta/2$ for sufficiently small time τ_z. Since we can always take K to be constant, we assume without loss of generality that:

$$\sup_{T \in [0, \tau_z[} \|K(T)\| < \infty. \tag{IV.3}$$

We now use the controller $u(t_k + T) = K(T) z^a(t_k)$, not with the state $z^a = (z_1^a, z_2^a, z_3^a)$ but with the estimate $\widehat{z^a} = (\widehat{z_1^a}, \widehat{z_2^a}, \widehat{z_3^a})$ obtained as explained in Section III. In other words, we use the controller:

$$u(t_k + T) = K(T) \widehat{z^a}(t_k). \tag{IV.4}$$

We denote the solution of (III.1)–(III.3) with the input $u = K(T)\widehat{z^a}$ by $\zeta^a(z^a, u, T)$ or, since $u = K(T)\widehat{z^a} = K(T)z^a + K(T)e^a$ for $e^a = \widehat{z^a} - z^a$, by $\zeta^a(z^a, e^a, T)$. Inequality (IV.2) now becomes:

$$\begin{aligned} V(\zeta_{12}^a(z^a, e^a, T)) - V(z_{12}^a) \quad \leq \quad & -\alpha_z T \|z_{12}^a\|^2 \\ & + \delta T \|e^a\|^2, \end{aligned} \tag{IV.5}$$

for some $\delta \in \mathbb{R}^+$ and for all $T \in [0, \tau_z[$. This inequality follows directly from the observation that linear controllers for linear systems result in a closed-loop system that is ISS with respect to the estimation error $e^a = \widehat{z^a} - z^a$. Inequality (IV.5) will be used in the next section to establish asymptotic stability for the unknown nonlinear system. Therefore, any linear controller resulting in an inequality of the form (IV.5) can be used.

V. STABILITY ANALYSIS

In this section we establish asymptotic stability of the unknown nonlinear system (II.1)-(II.2) when controlled by a linear control law (IV.4) enforcing inequality (IV.5) where $\widehat{z^a}$ is the state estimate obtained by the linear observer (III.4) enforcing inequality (III.5).

We start by clarifying the closed-loop model we will use in our analysis. For simplicity, we only discuss the case $n = 2$ although the results directly generalize to arbitrary $n \in \mathbb{N}$.

The *exact* and unknown sampled-data closed-loop model is described by:

$$\dot{z}_1^e = z_2^e \tag{V.1}$$

$$\dot{z}_2^e = z_3^e + bu(t_k) \tag{V.2}$$

$$\dot{z}_3^e = \frac{\partial z_3^e}{\partial z_1^e} z_2^e + \frac{\partial z_3^e}{\partial z_2^e}(z_3^e + bu(t_k)) \tag{V.3}$$

and valid for $t \in [t_k, t_k + \tau[$. The superscript e emphasizes the exactness of the model as opposed to the superscript a used for the approximate model (III.1)-(III.3). The input is constant and given by $u(t_k) = K(T)\widehat{z^a}(t_k)$. We denote the solution of this system by $\zeta^e(z^e(t_k), u(t_k), t)$ which is valid for $t \in [t_k, t_{k+1}[$. Since $u = K(T)\widehat{z^a} = K(T)z^e + K(T)e^e$, for the error $e^e = \widehat{z^a} - z^e$, we also denote $\zeta^e(z^e(t_k), u(t_k), t)$ by $\zeta^e(z^e(t_k), e^e(t_k), t)$.

The next result provides a bound for the error between the exact and approximate models.

Proposition 5.1: For any $\tau_m \in \mathbb{R}^+$ there exists a class $\mathcal{K}_\infty$ function ρ, depending on τ_m and the unknown nonlinear model, for which the following bound holds for all $T \in [0, \tau_m[$:

$$\begin{aligned} &\|\zeta^e_{12}(z, u, T) - \zeta^a_{12}(z, u, T)\| \\ \leq\ & T\rho(T)\left(\|z_{12}\| + \|u + z_3/b\|\right). \end{aligned} \tag{V.4}$$

Proof: A slight adaptation of the proof in Proposition 3.5 in [10] leads to the inequality:

$$\begin{aligned} &\|\zeta^e_{12}(z, u, T) - \zeta^{Euler}_{12}(z, u, T)\| \\ \leq\ & T\rho'(T)\left(\|(z_1, z_2)\| + \|u + z_3/b\|\right) \end{aligned} \tag{V.5}$$

where $\zeta^{Euler}(z, u, T)$ is the solution of the Euler approximation of (V.1)–(V.3) and $\rho'(T)$ is a class-$\mathcal{K}_\infty$ function. Since:

$$\zeta^{Euler}_{12}(z, u, T) = \begin{bmatrix} z_1 + Tz_2 \\ z_2 + T(z_3 + bu) \end{bmatrix}$$

we see that:

$$\begin{aligned} &\left\|\zeta^{Euler}_{12}(z, u, T) - \zeta^a_{12}(z, u, T)\right\| \\ =\ & \left\|\begin{bmatrix} -\frac{1}{2}T^2(z_3 + bu) \\ 0 \end{bmatrix}\right\| \\ \leq\ & dT^2\|u + z_3/b\|, \end{aligned} \tag{V.6}$$

for a suitable constant $d \in \mathbb{R}^+$. Combining inequalities (V.5) and (V.6) we arrive at:

$$\begin{aligned} &\|\zeta^e_{12}(z, u, T) - \zeta^a_{12}(z, u, T)\| \\ \leq\ & \|\zeta^e_{12}(z, u, T) - \zeta^{Euler}_{12}(z, u, T)\| \\ & + \left\|\zeta^{Euler}_{12}(z, u, T) - \zeta^a_{12}(z, u, T)\right\| \\ \leq\ & T\rho'(T)(\|z_{12}\| + \|u + z_3/b\|) \\ & + dT^2\|u + z_3/b\| \\ \leq\ & T\rho(T)(\|z_{12}\| + \|u + z_3/b\|), \end{aligned}$$

for $\rho(T) = \rho'(T) + dT$ and concludes the proof. ■

The main result of this paper is obtained by combining (V.4) with inequalities (III.5) and (IV.5).

Theorem 5.2: There exists a sampling time τ^* so that the linear control law (IV.4), designed based on the linear approximate model (III.1)–(III.3) and computed based on the state estimate $\widehat{z^a}$ given by the linear observer (III.4) designed for the linear approximate model, globally asymptotically stabilizes the nonlinear unknown model (II.1)–(II.2) satisfying assumptions 1) and 2).

Proof: We start by establishing the notation $\varepsilon^a(e^a(t_k), z^a(t_k), T)$ to denote the error $e^a(t_k + T)$ while emphasizing its dependence on $e^a(t_k)$ and $z^a(t_k)$, i.e.:

$$\begin{aligned} \varepsilon^a(e^a(t_k), z^a(t_k), T) &= e^a(t_k + T) \\ &= \widehat{\zeta^a}(e^a(t_k) - z^a(t_k), z^a(t_k), T) \\ &\quad -\zeta^a(z^a(t_k), e^a(t_k), T). \end{aligned} \tag{V.7}$$

Similarly, we introduce the notation $\varepsilon^e(e^e(t_k), z^e(t_k), T)$ to denote the error $e^e(t_k + T)$, i.e.:

$$\begin{aligned} \varepsilon^e(e^e(t_k), z^e(t_k), T) &= e^e(t_k + T) \\ &= \widehat{\zeta^a}(e^e(t_k) - z^e(t_k), z^e(t_k), T) \\ &\quad -\zeta^e(z^e(t_k), e^e(t_k), T). \end{aligned} \tag{V.8}$$

We now start the proof by rewriting inequality (V.4) using the error $e^e = \widehat{z^a} - z^e$. We note that $u - z^e_3/b = K(T)\widehat{z^a} - z^e_3/b = (K(T)z^e - z^e_3/b) + K(T)e^e$ and upon substitution in (V.4) we obtain:

$$\begin{aligned} &\|\zeta^e_{12}(z^e, u, T) - \zeta^a_{12}(z^e, u, T)\| \\ \leq\ & T\rho(T)\left(\|z^e_{12}\| + \|K(T)z^e - z^3_3/b\| + \|K(T)e^e\|\right) \\ \leq\ & T\rho(T)\left(\|z^e_{12}\| + c'\|z^e_{12}\| + \|K(T)e^e\|\right) \\ \leq\ & T\rho'(T)\left(\|z^e_{12}\| + \|e^e\|\right), \quad \forall T \in [0, \min\{\tau_z, \tau_m\}[\end{aligned}$$

where we used the fact that $K(T)z^e - z^3_3/b$ is a function of z^e_{12} only, given by $k_1(T)z^e_1 + k_2(T)z^e_2$, the constant c' is given by $c' = \sup_{T \in [0, \min\{\tau_z, \tau_m\}[} \|(k_1(T), k_2(T))\|$ (see assumption (IV.3)), and the $\mathcal{K}_\infty$ function ρ' is given by $\rho'(T) = \rho(T)\max\{1, c', \sup_{T \in [0, \min\{\tau_z, \tau_m\}[} \|K(T)\|\}$. By redefining the function ρ we arrive at:

$$\begin{aligned} &\|\zeta^e_{12}(z^e, u, T) - \zeta^a_{12}(z^e, u, T)\|^2 \\ \leq\ & T\rho(T)\left(\|z^e_{12}\|^2 + \|e^e\|^2\right). \end{aligned} \tag{V.9}$$

The remainder of the proof consists in showing the existence of $\tau^* \in \mathbb{R}^+$ for which:

$$W(z^e_{12}, e^e) = V(z^e_{12}) + \frac{\delta + \alpha_x}{\alpha_e} E(e^e)$$

becomes a Lyapunov function for the exact model (V.1)–(V.3) combined with the dynamics of e^e given by (V.8). The following long sequence of inequalities will be explained immediately thereafter.

$$\begin{aligned} & W(\zeta^e_{12}(z^e_{12}, e^e, T), \varepsilon^e(e^e, z^e_{12}, T)) - W(z^e_{12}, e^e) \\ =\ & W(\zeta^a_{12}(z^e_{12}, e^e, T), \varepsilon^a(e^e, z^e_{12}, T)) - W(z^e_{12}, e^e) \\ & + W(\zeta^e_{12}(z^e_{12}, e^e, T), \varepsilon^e(e^e, z^e_{12}, T)) \\ & - W(\zeta^a_{12}(z^e_{12}, e^e, T), \varepsilon^a(e^e, z^e_{12}, T)) \\ \leq\ & -\alpha_z T\|z^e_{12}\|^2 + \delta T\|e^e\|^2 - (\alpha_z + \delta)T\|e^e\|^2 \\ & + |W(\zeta^e_{12}(z^e_{12}, e^e, T), \varepsilon^e(e^e, z^e_{12}, T)) \\ & - W(\zeta^a_{12}(z^e_{12}, e^e, T), \varepsilon^a(e^e, z^e_{12}, T))| \qquad \text{(V.10)} \\ \leq\ & -\alpha_z T\|z^e_{12}\|^2 - \alpha_z T\|e^e\|^2 \\ & + \sigma\|\zeta^e_{12}(z^e_{12}, e^e, T) - \zeta^a_{12}(z^e_{12}, e^e, T)\|^2 \\ & + \sigma\|\varepsilon^e(z^e_{12}, e^e, T) - \varepsilon^a(z^e_{12}, e^e, T)\|^2 \qquad \text{(V.11)} \\ \leq\ & -\alpha_z T\|z^e_{12}\|^2 - \alpha_z T\|e^e\|^2 \\ & + \sigma T\rho(T)\left(\|z^e_{12}\|^2 + \|e^e\|^2\right) \\ & + \sigma\|\varepsilon^e(z^e, e^e, T) - \varepsilon^a(z^e, e^e, T)\|^2 \qquad \text{(V.12)} \\ =\ & -\alpha_z T\|z^e_{12}\|^2 - \alpha_z T\|e^e\|^2 \\ & + \sigma T\rho(T)\left(\|z^e_{12}\|^2 + \|e^e\|^2\right) \\ & + \sigma\|\zeta^e(z^e, e^e, T) - \zeta^a(z^e, e^e, T)\|^2 \qquad \text{(V.13)} \\ \leq\ & -\alpha_z T\|z^e_{12}\|^2 - \alpha_z T\|e^e\|^2 \\ & + 2\sigma T\rho(T)\left(\|z^e_{12}\|^2 + \|e^e\|^2\right) \qquad \text{(V.14)} \end{aligned}$$

Inequality (V.10) follows from (IV.5) and (III.5). Inequality (V.11) follows from the function $V^{\frac{1}{2}}$ being Lipschitz continuous with Lipschitz constant $\sigma_V^{\frac{1}{2}}$, function $E^{\frac{1}{2}}$ being Lipschitz continuous with Lipschitz constant $\sigma_E^{\frac{1}{2}}$, and $\sigma = \max\{\sigma_V, \sigma_E\}$. Inequality (V.13) follows from (V.7) and (V.8) while inequalities (V.12) and (V.14) result from a direct application of (V.9).

If we define:

$$\tau^* = \min\left\{\tau_e, \tau_z, \tau_m, \rho^{-1}\left(\frac{\alpha_z}{4\sigma d}\right)\right\}$$

it follows that (V.14) is upper bounded by:

$$-\frac{1}{2}\alpha_z T(\|(z_1^e, z_2^e)\|^2 + \|e^e\|^2)$$

for all T in the set $[t_k, t_k + \tau^*[$ thereby showing that W is a Lyapunov function proving asymptotic stability of $(z_1^e, z_2^e) = (0, 0)$ and $e^e = (0, 0, 0)$. ■

Although the discussion so far focused on the asymptotic stabilization problem, linearity of the model (III.1)–(III.3) allows us, with the same ease, to design controllers solving output regulation and tracking problems. The proof of Theorem 5.2 can easily be adapted so as to also apply to these cases.

VI. Experimental results

A. The experimental platform AMBER-3M

AMBER-3M is a planar, modular bipedal robot custom-built by AMBER Lab (see Fig. 1); here, modular refers to the fact that it has multiple leg designs that can be attached to test different walking phenomena [16]. It was previously used for the study of mechanics-based control [17]. In this particular study, we used a pair of lower limbs with point feet, which made AMBER-3M a 5-degree of freedom under-actuated walking robot. As shown in Fig. 1, the robot is connected to the world through a planner supporting structure, which eliminates the lateral motion but does not provide support to the robot in the sagittal plan.

B. Observer and controller design

In the first set of experiments we immobilized AMBER-3M while keeping one point foot in the air. The knee joint corresponding to the free standing point foot was then controlled by measuring the joint angle via an encoder. The torque commands produced by the data-driven controller were transformed, by the ELMO motor driver, into a torque applied at the knee joint by the BLDC motor. We modeled the controlled swinging lower limb as an inverted pendulum:

$$I\ddot{\theta} = u - mg\sin\theta$$

where I is the moment of inertia, m is the mass, g is gravity's acceleration, θ is the knee angle, and u is the input torque. Assumption 1) is satisfied since the relative degree of this system is $n = 2$. Assumption 2) is also satisfied since the supposedly unknown function $L_f^2 h = -mg\sin\theta$ is Lipschitz continuous, and the function $L_g L_f^1 h$ is indeed constant and given by $L_g L_f^1 h = b = 1/I = 2.442$.

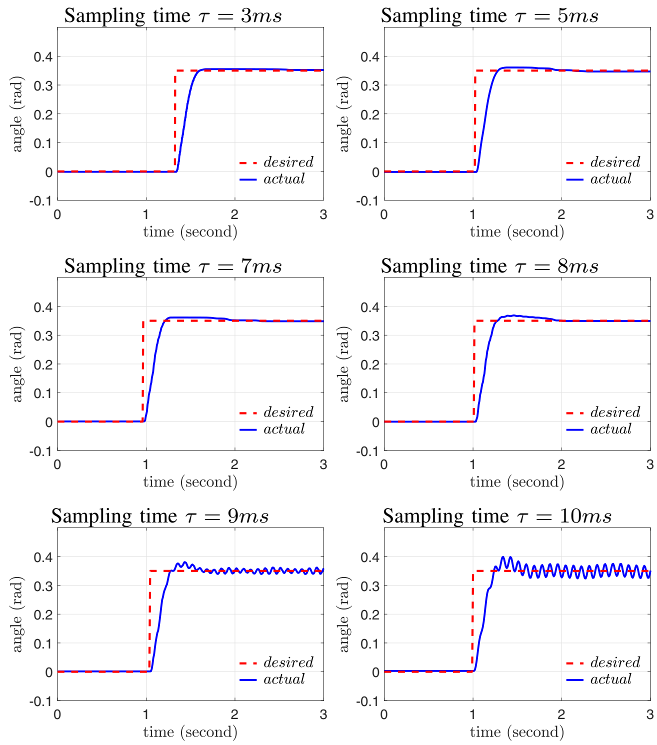

Fig. 2. Angle regulation to the desired set point of 0.35 rad for different values of the sampling time, τ, ranging from $3ms$ to $10ms$.

We designed the linear controller:

$$u = k_1 z_1 + k_2 z_2 - \frac{1}{b} z_3 \qquad \text{(VI.1)}$$

based on the model (III.1)–(III.3). The gains k_1 and k_2 were chosen so as to place the closed-loop eigenvalues at $e^{-\lambda\tau}$ with $\lambda = 20$. This resulted in:

$$k_1 = -\frac{e^{-2\lambda\tau}(e^{\lambda\tau} - 1)^2}{b\tau^2}, \qquad k_2 = \frac{-3 + 2e^{-\lambda\tau} + e^{-2\lambda\tau}}{2b\tau}.$$

This controller was then used with the estimate $\widehat{z^a}$ of z^a computed by a Luenberger observer whose gain was designed to place its eigenvalues at $e^{-m\lambda\tau}$ with $m = 3$. This lead to the gain matrix:

$$L = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T, \quad l_1 = 3 - 3e^{-m\lambda\tau}$$
$$l_2 = \frac{e^{-3m\lambda\tau}(e^{m\lambda\tau}-1)^2(5e^{m\lambda\tau}+1)}{2\tau}, \quad l_3 = \frac{e^{-3m\lambda\tau}(e^{m\lambda\tau}-1)^3}{\tau^2}.$$

C. Set-point regulation

Theorem 5.2 asserts the existence of a sufficiently small sampling time for which the previously described controller and observer stabilize the free standing point foot. In this experiment we decreased the sampling time, starting from $10ms$, until adequate performance for set-point regulation of the angle to the value of 0.35 rad was observed. Fig. 2 shows that adequate performance is achieved with sampling times smaller than or equal to 8ms. Based on these results, we used a sampling time of 5ms in all the other experiments.

To illustrate how the proposed control technique is robust to the value of b, assumed to be known, we repeated the

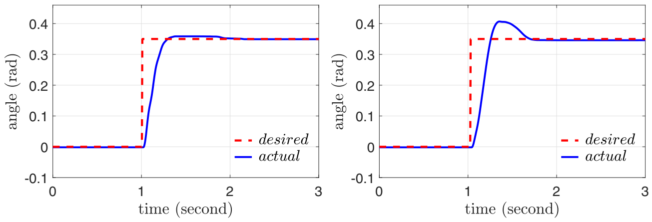

Fig. 3. Angle regulation to the desired set point of 0.35 rad when using an incorrect value for b. Left figures: $b = 1.8$, right figures: $b = 4$.

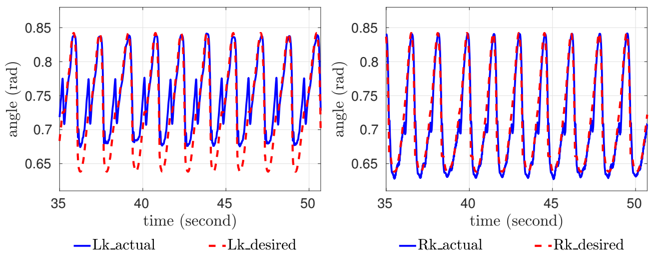

Fig. 4. Tracking performance on left (PD control) and right (data-driven control) for the knee joint while walking.

experiment with $\tau = 5ms$ and ranging b from 1.8 to 4 (the true value of b is 2.442). The top part of Figure 3 shows the evolution of the knee angle for $b = 1.8$ and $b = 4$. We observe that the regulation objective is still met although there is a larger overshoot and slower convergence for $b = 4$.

D. Comparison between the data driven controller and a PD controller

In this last experiment we compared the trajectory tracking capabilities of the data driven controller with a Proportional-Derivative (PD) controller that had been used in the past to implement walking gaits on AMBER-3M. This comparison was performed while the robot walked. Note that the pendulum model for the knee joint is no longer valid in the context of a walking gait where there are two distinct phases: the swinging phase where the lower limb does not make contact with the ground and the standing phase during which the weight of the whole robot is supported by the lower limb. Additionally, there are impacts due to foot strike (the robot is governed by a hybrid system model in this case). As shown in the movie [18], despite these considerations, we find that the data-driven controller performs well.

The tracking error associated with the PD controller can be seen in Fig. 4 by observing the difference between the actual and desired behavior. This figure also shows the tracking performance of the data-driven controller while the robot is locomoting, wherein it clearly outperforms the PD controller.

VII. CONCLUSIONS

The results described in this paper are but a first step towards a general purpose data driven control methodology. The authors are currently working on several extensions such as: 1) relaxing feedback linearizability to partial feedback linearizability (this will require making certain assumptions on the residual and zero-dynamics); 2) identifying the function $L_g L_f^{n-1} h$ from data (this is a classical and hard adaptive control problem that becomes simpler when the sign of $L_g L_f^{n-1} h$ is known since we can use, e.g., Lyapunov based controller based adaptive controllers based on the Immersion and Invariance approach [19]). Also under investigation are the robustness properties of the proposed control methodology, especially in what regards sensor noise. Although it is unavoidable that model-free controllers are more sensitive to noise than model-based controllers, since all the model information needs to be extracted from data, it is important to quantify such sensitivity.

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